PHYS 813: Statistical Mechanics, Assignment 8

Due 5/14/09

1. Consider a Bose gas of photons in a volume V and temperature T. The photon energy is $\epsilon = pc$ where p is the photon momentum and c is the speed of light. For photons, the chemical potential μ is always $\mu = 0$. The number of photons is not well determined since the number of photons at $\epsilon = 0$ cannot be defined (cf. the BE expression). Assume that the initial temperature is high and there are intially N photons with nonzero energies.

   (a) Find an expression for the maximum value of N at a given T.

   (b) For a given N, the lowest temperature "supporting" this number can be loosely called T_c, as if the temperature is lowered below T_c, some photons "condense". Calculate the pressure p and the energy U for $T < T_c$. Is the classical relation $p = U/3V$ preserved?

2. Consider a noniteracting Bose gas in which the lowest excited state is separated from the ground state by a gap Δ, i.e.,

$$\epsilon(\boldsymbol{p}) = \begin{cases} 0 & p = 0 \\ \Delta + p^2/2m & p \neq 0 \end{cases}$$

   where $p = |\boldsymbol{p}|$ and m is the mass of a particle. We can assume that the spectrum above the gap is quasicontinuous. Find the temperature dependence of the specific heat C_V for low temperatures T, i.e., for $kT \ll \Delta$, where k is the Boltzman constant.

3. Atoms in solids can exhibit various collective vibrational motions. The normal frequency ω of a given vibrational mode is connected with the wave number k of a corresponding plane-wave motion by the so-called dispersion relation $\omega = Ak^s$, where A is a constant. For the best-known phonon waves, the relation is $\omega = vk$, where v is the velocity of sound in the solid. Show that the vibrational motion contribution to the specific heat of a solid at low temperatures T is proportional to $T^{3/s}$. For phonons, this dependence is called the Debye's T^3 law. The case of $s = 2$ corresponds to the spin waves propagating in ferromagnetic systems. Use Debye's assumption that the frequencies are distributed continuously from zero to a cutoff value ω_{D} and that the density of states in k is $g(k) = 12\pi V k^2/(2\pi)^3$.

4. Despite its simplicity, the free-electron model of the electron conduction in metals predicts various properties quite successfully. Use this model to calculate the isothermal compressibility of a metal as a function of the electron number density n and the Fermi energy ϵ_F at $T = 0$ K. Assume that the electron gas is completely degenerate, i.e., that the occupation number is a Heaviside step function, and it occupies a constant volume V at temperature T.

   (a) Derive expressions for the total number of electrons N and energy U.

   (b) Derive an expression for pressure p as a function of U and V.

   (c) Compute the isothermal compressibility $\kappa = -(1/V)(\partial V/\partial p)_T$.

5. Consider a degenerate (assume here $T = 0$), relativistic (i.e., the relation between particle's energy ϵ and momentum p is $\epsilon = pc$, where c is the speed of light) gas of fermions of spin $\frac{1}{2}$.

   (a) Find the expression for the total energy of the system as a function of the expectation number of particles N and the total volume V.

   (b) Calculate the pressure in this systems.

6. Suppose that in some sample the density of states of electrons $g(\epsilon)$ is a constant g_0 for energies $\epsilon > 0$ ($g(\epsilon) = 0$ for $\epsilon < 0$) and the total number of electrons is N.

(a) Calculate the Fermi energy ϵ_F at $T = 0$ K.

(b) For nonzero temperatures, derive the condition that the system is nondegenerate. Express your answer in terms of the relation between kT and ϵ_F.

(c) Show that the electronic specific heat is proportional to T when the system is nearly degenerate. Use simple physical arguments rather than elaborate derivations.

7. Since in the Fermi-Dirac statistics the grand partition functions for ideal gas is given by $\mathcal{Z}(V, T, \mu) = \prod_i \left(1 + z\mathrm{e}^{-\beta\epsilon_i}\right)$, where $\beta = 1/kT$, ϵ_i are single-particle energy levels, μ is the chemical potential, and $z = \mathrm{e}^{\mu/kT}$, this form does not exhibit a direct relation to the macroscopic parameters of the gas, as it is the case for its classical counterpart. Show that the relation

$$q(V, T, \mu) = \ln \mathcal{Z}(V, T, \mu) = \frac{pV}{kT} \tag{1}$$

does hold for the Fermi-Dirac ideal gas, proceeding as follows:

(a) Using the relations $q = (gV/\lambda^3)f_{5/2}(z)$ and $N = (gV/\lambda^3)f_{3/2}(z)$, where g is the spin degeneracy, $\lambda = h/\sqrt{2\pi mkT}$, and

$$f_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}dx}{z^{-1}\mathrm{e}^x + 1},$$

with $\Gamma(n) = (n-1)(n-2)\cdots\frac{1}{2}\sqrt{\pi}$ for half integer arguments, show that

$$N = z\left(\frac{\partial q}{\partial z}\right)_{V,T}.$$

(b) Starting from $U = \sum n_i\epsilon_i$, where $n_i = 1/(z^{-1}\mathrm{e}^{\beta\epsilon_i} + 1)$ is the occupation number for state ϵ_i, derive an expression for U which can be related to q.

(c) Show that the differential of $q(V, T, \mu)$ can be written as $dq = -Ud\beta - \beta đW - Nd\alpha$, where W is work performed on the gas and $\alpha = -\ln z$.

(d) Show that this relation implies that the entropy $S = k(q + \alpha N + \beta U)$.

(e) Combine the derived relation with the Euler formula to get Eq. (1).