PHYS 620: Exam III

November 22, 2024, 55 minutes, closed book.

Please start each solution on a fresh sheet of paper, use only one side of the paper. Try to show how well you understand the problems. Always justify your reasoning.

1. Show that the validity of Kepler's first two laws for any body orbiting the sun implies that the force $\boldsymbol{F}$ (assumed conservative) of the sun on any body is central and proportional to $1/r^2$, where r is the distance from the sun to the planet (this is experimental discovery of the law of gravity). *Hints:* The first Kepler law states that orbits are ellipses, so that that $r(\phi) = c/(1 + \epsilon \cos\phi)$, where ϕ is the polar angle, while c and ϵ are constants. The second law states that the area A swept per time, $dA/dt = l/2\mu$, where l is the magnitude of angular momentum and μ is the reduced mass of the sun–planet system. First use the second law to prove that the force is central. Next you can use Newton's in the form

$$[u''(\phi) + u(\phi)]\,\frac{l^2 u^2(\phi)}{\mu} = -F,$$

where $u = 1/r$ and primes denote derivatives with respect to ϕ, to determine the $1/r^2$ dependence.

2. I am standing (wearing crampons) on a perfectly frictionless flat merry-go-round, which is rotating counterclockwise with angular velocity Ω about its vertical axis. (a) I am holding a puck at rest just above the floor (of the merry-go-round) and release it. Describe qualitatively the puck's path as seen from above by an observer who is looking down from a nearby tower (fixed to the ground) and then as seen by me on the merry-go-round. In the latter case explain what I see in terms of the centrifugal and Coriolis forces. (b) Answer the same questions for a puck which is released from rest by a long-armed spectator who is standing on the ground leaning over the merry-go-round. *Hints*: Assume that Earth is an inertial system. You can use without proof Newton's equation in the rotating frame: $m\ddot{\boldsymbol{r}}' = \boldsymbol{F} + 2m\dot{\boldsymbol{r}}' \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \boldsymbol{r}') \times \boldsymbol{\Omega}$.

3. Derive the form of the inertia tensor computed in the coordinate frame S defined by the principal axes of inertia of a solid body. Express your answer in terms of the principal moments of inertia of this body. *Hints:* Consider first a pointwise body of mass m and show that its tensor of inertia in an inertial frame S_0, defined as $I_{ii} = m(r^2 - x_i^2),\ I_{ij} = -m x_i x_j$ for $i \neq j$, can be expressed as

$$\mathbb{I} = m\left([r]^T [r] \mathbb{1} - [r]\,[r]^T\right),$$

where $[r] = [x_1\ x_2\ x_3]^T$. Then write an analogous expression for the tensor of inertia in frame S and transform it to the desired form. You can use without proof the following relation between coordinates of a vector in S_0 and S:

$$[r'] = \mathbb{S}[r],$$

where $\mathbb{S}$ is the matrix whose rows are the eigenvectors of $\mathbb{I}$ in S_0.