PHYS 620: Exam II

October 25, 2024, 55 minutes, closed book.

Please start each solution on a fresh sheet of paper, use only one side of the paper. Try to show how well you understand the problems. Always justify your reasoning.

1. A pendulum is made from a massless spring of force constant k and unstretched length l_0. The spring is suspended on one end from a fixed pivot O and has a mass m attached to its end. The spring can compress and stretch but cannot bend. The whole system is confined to a single vertical plane. (a) Write down the Lagrangian ($\mathcal{L} = T - U$ is the difference of kinetic and potential energies) for the pendulum using as the generalized coordinates q_i the polar coordinates: the angle ϕ and the length of the spring r; (b) Find the two Lagrange equations of motion ($\partial\mathcal{L}/\partial q_i - d/dt(\partial\mathcal{L}/\partial\dot{q}_i) = 0$); ($c$) Solve these equations in the limit of small departures ϵ and ϕ from the equilibrium position r_0 and the equilibrium angle $\phi = 0$, respectively. Keep in your equations only terms linear in ϵ or ϕ and their derivatives (thus, drop also terms containing the products of ϵ and ϕ or their derivatives).

2. Find Euler-Lagrange equations resulting from the minimization of the functional depending on the function $\boldsymbol{y}(x) = (y_1(x), y_2(x))$
$$J[\boldsymbol{y}] = \int_{x_1}^{x_2} f(\boldsymbol{y}(x), \boldsymbol{y}'(x), x)dx = \int_{x_1}^{x_2} f(y_1(x), y_2(x), y_1'(x), y_2'(x), x)$$
with a constraint equation $g(\boldsymbol{y}(x), x) = 0$. ($a$) By writing $\boldsymbol{y}(x)$ in the form $\boldsymbol{y}(x) = \boldsymbol{y}_0(x) + \alpha\boldsymbol{\eta}(x)$, where $\boldsymbol{y}_0(x)$ minimizes the functional and $\boldsymbol{\eta}(x) = (\eta_1(x), \eta_2(x))$, with $\eta_i(x)$ an arbitrary fixed function except that $\eta_i(x_j) = 0$, change the functional J into a function of α and calculate its derivative at $\alpha = 0$; (b) Use integration by parts to remove the derivatives of $\boldsymbol{\eta}$ and explain why the resulting equation does not lead to Euler-Lagrange equations; (c) Use the substitution $\boldsymbol{y}(x) = \boldsymbol{y}_0(x) + \alpha\boldsymbol{\eta}(x)$ in the constraint equation, compute the derivative of this equation with respect to α, and argue that you can subtract the resulting equation multiplied by an arbitrary function $\lambda(x)$ (called an undetermined Lagrange's multiplier) from the equation derived in point (b); (d) Argue that by a proper choice of $\lambda(x)$ one can remove the difficulty encountered in point (b) and derive the Euler-Lagrange equations with undetermined Lagrange's multiplier.

3. Consider a wedge of mass m_2 with the incline angle α and height h sliding without friction on a horizontal plane in Earth's gravitational field with a constant gravity acceleration g. There is a small block of mass m_1 on the incline, sliding without friction. Assume that at time zero the system is at rest with the small block at the top of the wedge. Determine the motion of both bodies as a function of time (use the generalize coordinates $q_1(t)$—the distance of the small block from the top of the edge and $q_2(t)$—the distance of the vertical wall of the edge from an arbitrary point on the plane). Find Hamilton's equations of motion. *Hints*: $\mathcal{H} = T + U$; $p_i = \partial\mathcal{L}/\partial\dot{q}_i$; $\dot{p}_i = -\partial\mathcal{H}/\partial q_i$; $\dot{q}_i = \partial\mathcal{H}/\partial p_i$. Proceed as follows: ($a$) Write $\mathcal{H}$ in Cartesian coordinates and transform to generalize coordinates; (b) Calculate generalized momenta; (c) Express generalized velocities by generalize momenta; (d) Substitute generalized velocities by generalized momenta in $\mathcal{H}$. You may skip the algebra and use the result
$$\mathcal{H} = \frac{1}{m_2 + m_1\sin^2\alpha}\left(\frac{m_1 + m_2}{2m_1}p_1^2 + \frac{1}{2}p_2^2 - p_1p_2\cos\alpha\right) + m_1g(h - q_1\sin\alpha);$$
(e) Now write down Hamilton's equations of motion.