PHYS 620: Exam I

September 27, 2024, 55 minutes, closed book.

Please start each solution on a fresh sheet of paper, use only one side of the paper. Try to show how well you understand the problems. Always justify your reasoning.

1. A particle of mass m slides without friction on a horizontal table. The particle is attached to one end of a massless spring of equilibrium length a and spring constant k. The other end of the spring is attached to a point on the table, such that the spring can rotate around this point without friction. The derivatives of vector $\boldsymbol{r}$ in cylindrical coordinates are

$$\dot{\boldsymbol{r}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\boldsymbol{z}}; \quad \ddot{\boldsymbol{r}} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\boldsymbol{z}}$$

   (a) What is the net force acting on the particle? Note that the gravity force is balanced by the reaction of the table and there is no friction. Is the net force conservative? *Hint*: you do not need to compute $\boldsymbol{\nabla} \times \boldsymbol{F}$.

   (b) Find the expression for the kinetic energy T, the potential energy U (derive it from the work integral), and the angular momentum $\boldsymbol{l}$ of the particle. Show that $\boldsymbol{l}$ is constant in time.

   (c) Define the effective potential energy U_{eff} as the sum of U and the part of T depending on $|\boldsymbol{l}|$. Sketch the U and U_{eff} potential energy functions. Why such representation of the problem is useful?

   (d) Derive the condition for the particle to move exactly on a circular orbit of radius r_0 with a constant angular velocity ω_0, where $\omega = \dot{\phi}$, and find ω_0. Note that ω_0 is not equal to $\sqrt{k/m}$.

2. Find the work performed by the force $\boldsymbol{F} = k[x^2, xy, -xz]$, where k is a constant, on the path along the great circle of a sphere that is span by points $\boldsymbol{r}_1 = R[1, 1, 0]/\sqrt{2}$ and $\boldsymbol{r}_2 = R[0, 0, 1]$, where R is the radius of the sphere (choose the shorter of the two possible paths). [The great circle considered is the intersection of the sphere with the plane $x - y = 0$.] Will this work be equal to the difference of potential energies between the two points? Under what conditions will it be equal to the difference of kinetic energies (do not check this condition).

3. Consider the problem of finding the function $y(x)$ which makes the value of the following functional to be an extremum

$$J[y] = \int_{x_1}^{x_2} f(y(x), y'(x), x)dx = \text{ extremum},$$

   where the function f and $y(x_i) = y_i$ are known. Show that such function can be found by solving a differential equation for $y(x)$. Find this equation.