PHYS 620: Final Exam

December 16, 2024, 120 minutes, closed book.

Please start each solution on a fresh sheet of paper, use only one side of the paper. Try to show how well you understand the problems. Always justify your reasoning.

1. A particle is confined to move on the surface of a circular cone in Earth's gravitational field with the cone's axis on the vertical z axis, vertex at the origin (pointing down), and half-angle α. (a) Find the equation of constraints and a generalized coordinate system satisfying this equation by making one variable constant. Write down the Lagrangian $\mathcal{L}$ first in terms of Cartesian coordinates and then derive from this its form in terms of the generalized coordinates. (b) Find the two equations of motion. Derive the expression for the angular momentum $\boldsymbol{l}$ of the particle and interpret one of the equations in terms of l_z. Then use the relation thus obtained to eliminate a variable from the other equation in favor of l_z. Does this equation make sense in the case that $l_z = 0$? Find the values of coordinates at which the particle can remain in a horizontal circular path. (c) Suppose that the particle on such circular path is given a small kick, so that the motion is now described as $r(t) = r_0 + \epsilon(t)$, where r is the distance from the origin, r_0 is the value of r on the circular path, and $\epsilon(t)$ is small. Determine whether the circular path is stable. If so, what kind of motion is that and what are its characteristic parameters?

2. Noether's theorem asserts a connection between invariance of Lagrangian and conservation laws. In particular, translational invariance of the Lagrangian implies conservation of total linear momentum and time invariance conservation of energy. Prove that rotational invariance of $\mathcal{L}$ implies conservation of total angular momentum. Suppose that the Lagrangian of an N-particle system whose potential energy depends only on positions of particles is unchanged by rotations about a certain axis. (a) Without loss of generality, take this axis to be the z axis and use spherical coordinates to describe the position of each particle. Then the invariance considered means that the Lagrangian is unchanged when all of the particles are simultaneously moved from (r_i, θ_i, ϕ_i) to $(r_i, \theta_i, \phi_i + \epsilon)$ (same ϵ for all particles). Hence show that

$$\sum_i^N \frac{\partial \mathcal{L}}{\partial \phi_i} = 0$$

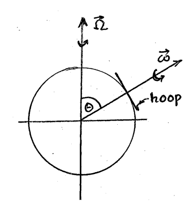
(b) Use Lagrange's equations to show that this implies that the z-component of the total angular momentum, L_z, is constant. *Hints:* You may use without proof the the expression for kinetic energy in spherical coordinates: $(m/2)(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2)$.

3. Consider two-body problem with a conservative central force between pointwise particles of masses m_1 and m_2. One can show that the equation of motion in the relative coordinate vector $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ is

$$\mu\ddot{\boldsymbol{r}} = -\frac{dU(r)}{dr}\hat{\boldsymbol{r}}$$

where $\mu = m_1 m_2/(m_1 + m_2)$ and $U(r)$ is the potential energy. (a) Show that the motion of $\boldsymbol{r}$ is restricted to a plane. (b) Using Lagrange's formalism, derive equations of motion in this plane using polar coordinates ρ and ϕ [you can use without proof the following form of the Lagrangian: $\mathcal{L} = (m_1 + m_2)\dot{\boldsymbol{R}}^2/2 + \mu\dot{\boldsymbol{r}}^2/2 - U(r)$, where $\boldsymbol{R}$ is the center of mass vector]. (c) Show that the equation in ρ can be rewritten as $\mu\ddot{\rho} = -(d/d\rho)U_{\text{eff}}(\rho)$. Find the explicit expression for U_{eff}.

4. The Coriolis force can produce a torque on a spinning object. To illustrate this, consider a horizontal hoop of mass m and radius r_0 spinning counterclockwise with angular velocity ω about its vertical axis at colatitude θ, see the figure. Show that the Coriolis force due to the earth's rotation produces a torque of magnitude $m\omega\Omega r_0^2 \sin\theta$ directed to the west, where Ω is the earth's angular velocity. Neglect the centrifugal force and assume that the vertical axis of the hoop is directed towards the center of the Earth. (This torque is the basis of the gyrocompass.) You can use without proof Newton's equation in the rotating frame: $m\ddot{\boldsymbol{r}}' = \boldsymbol{F} + 2m\dot{\boldsymbol{r}}' \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \boldsymbol{r}') \times \boldsymbol{\Omega}$.



5. A rigid body of an irregular shape and tensor of inertia $\mathbb{I}$ rotates around a fixed point $\mathcal{O}$ with angular velocity $\boldsymbol{\omega}$ subject to a torque $\boldsymbol{\Gamma}$. Find the equations of motion for the components of $\boldsymbol{\omega}$ in a suitably chosen body-fixed reference frame $\mathcal{S}_1$ centered at $\mathcal{O}$ (Euler's equations). Start from the Newton's rotational equation in space-fixed coordinate system $\mathcal{S}_0$ also centered at $\mathcal{O}$. *Hints:* For any vector, the derivatives in $\mathcal{S}_0$ and $\mathcal{S}_1$ are related by

$$\left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathcal{S}_0} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{\mathcal{S}_1} + \boldsymbol{\omega} \times \boldsymbol{A}$$

To use this equation, you have to fits find the components of the angular momentum vector in $\mathcal{S}_1$.

6. (a) Find the normal frequencies for the system of two carts and three springs shown in the figure, for the case that $m_1 = m_2$ and $k_1 = k_3$, (but k_2 may be different). Check that your answer is correct for the case that $k_1 = k_2$ as well. (b) Find and describe the motion in each of the two normal modes in turn.