PHYS 620: Assignment 6

Due 10/11/24

1. Recall the proof of Lagrange's equations for a single particle constrained to move on a two-dimensional surface. Go through the same steps to prove Lagrange's equations for a system consisting of two particles subject to various unspecified constraints. [*Hints*: The net force on particle 1 is the sum of the total constraint force $\boldsymbol{F}_1^{\text{constr}}$ and the total nonconstraint force $\boldsymbol{F}_1$, likewise for particle 2. The constraint forces come in many guises (the normal force of a surface, the tension force of a string tied between the particles, etc.), but it is always true that the net work done by all constraint forces in any displacement consistent with the constraints is zero — this is the defining property of constraint forces. Assume that the nonconstraint forces are conservative and therefore derivable from a potential energy $\boldsymbol{F}_i = -\boldsymbol{\nabla}_i U(\boldsymbol{r}_1, \boldsymbol{r}_2, t)$. Write down the action integral representing the paths as $\boldsymbol{r}_i(t) = \boldsymbol{r}_{i0}(t) + \alpha\boldsymbol{\epsilon}_i(t)$, where $\boldsymbol{r}_{i0}(t)$ are the paths satisfying Newton's equations and α is a number. Paralleling the proof for single particle, show that the devrivative of the action integral with respect to α reduces to an expression that is zero by the defining property of constraint forces.]

2. Using the method of undetermined Lagrange multipliers, find the dimensions of the rectangular box of maximum volume circumscribed by

   (a) A sphere of radius R.

   (b) An ellipsoid with semiaxes a, b, and c.

3. The method of Lagrange multipliers works perfectly well with non-Cartesian coordinates. Consider a mass m that hangs from a string, the other end of which is wound several times around a wheel (radius R, moment of inertia I) mounted on a frictionless horizontal axle. Use as coordinates for the mass and the wheel x, the distance fallen by the mass, and ϕ, the angle through which the wheel has turned (both measured from some convenient reference position). Write down the Lagrange equations with multipliers for these two variables and solve them (together with the constraint equation, $f(x, \phi) = 0$) for $\ddot{x}$, $\ddot{\phi}$, and the Lagrange multiplier. Write down Newton's equations for the mass and wheel, and use them to check your answers for $\ddot{x}$ and $\ddot{\phi}$. Show that $\lambda\partial f/\partial x$ is indeed the tension force on the mass. Comment on the quantity $\lambda\partial f/\partial\phi$.

4. Consider the simple pendulum with the length l and the mass of the bob m. Use the coordinate system with $\hat{\boldsymbol{x}}$ pointing down. Do not assume small angles. (a) Write down the Lagrangian for the system and the equation of the constraint in terms of the

Cartesian coordinates x and y. Then eliminate the y variable and write the Lagrange equation (do not solve it). Only now introduce polar coordinates and show that this equation is the same as derived earlier in Newton's formalism. (*b*) Now do not eliminate one variable but instead write down the set of Lagrange equations in x and y with the undetermined multiplier and then obtain from them a differential equation involving only $x(t)$ and its derivatives. Compare this procedure with that of point (a). (*c*) For comparison, write down Newton's equations in variables x and y and reduce them to a single equation involving only $x(t)$ and its derivatives. Do not use polar coordinates here. (*d*) Now redo points (a) and (c) introducing polar coordinates from the beginning.

5. Consider a double Atwood machine, in Earth's gravitational field, constructed as follows. A mass m_1 is suspended from a string that passes over a massless pulley on frictionless bearings. The other end of this string supports a second similar pulley, over which passes a second string supporting a mass of m_2 at one end and m_3 at the other end. Using as the coordinates the vertical positions of the three masses, write down the Lagrange equations of motion with an undetermined Lagrange multiplier. Use these equations to find the tensions in both moving strings.

6. Consider a wedge of mass m_2 with the incline angle α and height h sliding without friction on a horizontal plane in Earth's gravitational field with a constant gravity acceleration g. There is a small block of mass m_1 on the incline, sliding without friction. Assume that at time zero the system is at rest with the small block at the top of the wedge. Determine the motion of both bodies as a function of time (use any convenient coordinates but express your final answer in terms of $q_1(t)$—the distance of the small block from the top of the edge and $q_2(t)$—the distance of the vertical wall of the edge from an arbitrary point on the plane). Find the time it takes for the block to reach the bottom of the incline.

   (a) Use Newton's formalism. *Hint:* you do not have to directly solve Newton's equations to get the required answer.

   (b) Use Lagrange's equations.

   (c) Use Lagrange's equations with Lagrange's multipliers.

   (d) If you have not done so before, write Newton's equations for this system explicitly specyfying the forces.