PHYS 620: Assignment 5

Due 10/4/2024

1. (*a*) Write down the Lagrangian $\mathcal{L}(x_1, x_2, \dot{x}_l, \dot{x}_2)$ for two particles of equal masses, $m_1 = m_2 = m$, confined to the x axis and connected by a spring with potential energy $U = -(1/2)kx^2$. [Here x is the extension of the spring, $x = (x_1 - x_2 - l)$, where l is the spring's unstretched length. Assume that mass 1 remains to the right of mass 2 at all times.] (*b*) Rewrite $\mathcal{L}$ in terms of the new variables $X = (x_1 + x_2)/2$ (the CM position) and x (the extension), and write down the two Lagrange equations for X and x. (c) Solve for $X(t)$ and $x(t)$ and describe the motion.

2. A frictionless rectangular plate rotates around one of its edges with a constant angular velocity $\dot{\theta}(t) = \omega$ in Earth's gravitational field, with the axis of rotation placed horizontally. There is a bar of soap (mass m) on the plate. At time $t = 0$, the bar is at rest and the plane is horizontal. For $t > 0$, the soap will start to slide toward the downhill edge. Show that the equation of motion for the soap has the form $\ddot{x} - \omega^2 x = -g \sin \omega t$, where x is the soap's distance from the downhill edge. Solve this for $x(t)$, given that $x(0) = x_0$. [You can easily solve the homogeneous equation; for a particular solution try $x = A \sin \omega t$ and solve for A.] Find an expression for the normal force acting on the object as a function of time.

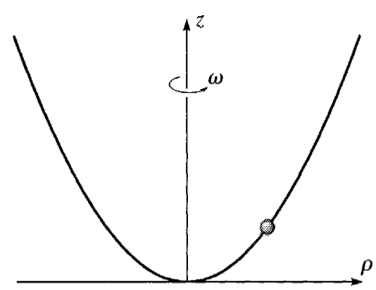
3. A particle of mass m can slide freely along a straight wire placed in the $x - y$ plane whose perpendicular distance to the origin O is h. Denote the projection of O on the wire by C. The line OC rotates around the origin (in $x - y$ plane) at a constant angular velocity ω. The particle is subject to a gravitational force acting down the y axis. Find the equations of motion under the initial conditions $\theta(0) = 0$, $q(0) = 0$, and $\dot{q}(0) = 0$, where θ is the polar angle of OC and q is the distance of the particle from C. Sketch the solution.

4. Two equal masses, $m_l = m_2 = m$, are joined by a massless string of length L that passes through a hole in a frictionless horizontal table. The first mass slides on the table while the second hangs below the table and moves up and down in a vertical line. (*a*) Assuming the string remains taut, write down the Lagrangian for the system in terms of the polar coordinates (r, ϕ) of the mass on the table. (*b*) Find the two Lagrange equations and interpret the ϕ equation in terms of the angular momentum l of the first mass. (*c*) Express $\dot{\phi}$ in terms of l and eliminate $\dot{\phi}$ from the r equation. Now use the r equation to find the value $r = r_0$ at which the first mass can move in a circular path. Interpret your answer in Newtonian terms. (*d*) Suppose the first mass is moving in this circular path and is given a small radial nudge. Write $r(t) = r_0 + \epsilon(t)$ and rewrite the r equation in terms of $\epsilon(t)$ dropping all powers of $\epsilon(t)$ higher than

linear. Show that the circular path is stable and that $r(t)$ oscillates sinusoidally about r_0 . What is the frequency of its oscillations?

5. A particle is confined to move on the surface of a circular cone with its axis on the vertical z axis, vertex at the origin (pointing down), and half-angle α. (a) Write down the Lagrangian $\mathcal{L}$ in terms of the spherical polar coordinates r and ϕ. (b) Find the two equations of motion. Interpret the ϕ equation in terms of the angular momentum l_z, and use it to eliminate $\dot{\phi}$ from the r equation in favor of the constant l_z. Does your r equation make sense in the case that $l_z = 0$? Find the value r_0 of r at which the particle can remain in a horizontal circular path. (c) Suppose that the particle is given a small radial kick, so that $r(t) = r_0 + \epsilon(t)$ where $\epsilon(t)$ is small. Use the r equation to decide whether the circular path is stable. If so, with what frequency does r oscillate about r_0?

6. Consider a bead of mass m sliding without friction on a wire that is bent in the shape of a parabola and is being spun with constant angular velocity w about its vertical axis, as shown on the figure. Use cylindrical polar coordinates and let the equation of the parabola be $z = k\rho^2$. Write down the Lagrangian in terms of ρ as the generalized coordinate. Find the equation of motion of the bead and determine whether there are positions of equilibrium, that is, values of ρ at which the bead can remain fixed, without sliding up or down the spinning wire. Discuss the stability of any equilibrium positions you find.



7. A uniform density cube of mass m, side $2b$, and center at C is placed on a *fixed* horizontal cylinder of radius r and center O in Earth's gravitational field. The cube is originally put so that C is centered above O, but it can roll from side to side *without slipping*. Use the Lagrangian approach to find the angular frequency of small oscillations about the top position. Make the small angle approximation at the level of the Lagrangian and retain only the leading term separately in the kinetic and potential energy. *Hints:* The kinetic energy of the cube consists of the kinetic energy of the center of mass and of the kinetic energy of rotational motion of the cube around its center of mass with the moment of intertia of the cube $I = 2mb^2/3$. Find x and y coordinates of COM from the triangles marked on the second figure. For the small angle approximation: $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$.