PHYS 620: Assignment 4

Due 9/27/2024

1. Consider the problem of brachistochrone. A car moves on a track between points 1 and 2, with 1 higher above the ground. The car is launched from point 1 with a fixed speed v_0. Use the coordinate system such that x is directed vertically down and y is horizontal.

   (a) Use the energy conservation to find an expression for the velocity as a function of x.

   (b) By comparison with the derivation for the case $v_0 = 0$ performed in the textbook, show that the solution for x is shifted by $v_0^2/2g$ and the expression for y is the same as for the case $v_0 = 0$. There is no need to repeat the whole derivation.

   (c) Show that for the identical values of the free parameters in the solution, the $v_0 \neq 0$ $x(y)$ curve is just shifted up by $v_0^2/2g$ compared to the $v_0 = 0$ curve.

   (d) Now assume that the initial and final points are the same for the two cases. One can take $x_1 = y_1 = 0$ without loss of generality. Write down the system of equations for the free parameters of the trajectory. Show that these equations can be solved, but do not go over the solution. Can the radius of the cycloid be the same in both cases? Draw qualitatively the curves for the two cases and interpret physically.

2. Consider a right circular cylinder of radius R centered on the z axis. Find the equation giving ϕ as a function of z for the geodesic (shortest path) on the cylinder between two points with cylindrical polar coordinates (R, ϕ_1, z_1) and (R, ϕ_2, z_2). Describe the geodesic. Is it unique? By imagining the surface of the cylinder unwrapped and laid out flat, explain why the geodesic has the form it does.

3. A surface of revolution is generated as follows: Two fixed points (x_1, y_1) and (x_2, y_2) in the x, y plane are joined by a curve $y = y(x)$. [Actually, you will make life easier if you start out writing this as $x = x(y)$.] The whole curve is now rotated about the x axis to generate a surface. Show that the curve for which the area of the surface is minimum has the form $y = y_0 \cosh[(x - x_0)/y_0]$, where x_0 and y_0 are constants. (This is often called the soap-bubble problem, since the resulting surface is usually the shape of a soap bubble held by two coaxial rings of radii y_l and y_2.)

4. A double pendulum consists of two simple pendula, with one pendulum suspended from the bob of the other. Assume that both pendula have equal lengths, have bobs of equal mass, and are confined to move in the same plane. Find the Lagrange equations of motion for this system. Do not assume small angles. Do not solve these equations.

5. A particle of mass m moves in a plane under the influence of a central force $\boldsymbol{F} = -Ar^{\alpha-1}\hat{\boldsymbol{r}}$, where r is the distance from the origin, with constant A and α ($\alpha > 0$ and $\alpha \neq 1$). Assume the potential energy to be zero at the origin of the coordinate system. In items (a) and (b) $\boldsymbol{F}$ is the only force acting on the particle.

   (a) Assume appropriate generalized coordinates and find Lagrange's equations of motion.

   (b) Show that these equations imply that the angular momentum about the origin and the total energy are conserved.

   (c) Now assume that in addition there is a uniform force of gravity acting in the plane (down the vertical y axis). Find Lagrange's equations of motion for this case.

   (d) Are the angular momentum about the origin and the total energy still conserved?

6. A particle of mass m is constrained to move on a circle of radius R. The circle rotates in its plane about one point on the circle with a constant angular speed ω. The position of the particle relative to the circle can be described by an angle ϕ that the particle's radius (from the origin of the circle) forms with the circle's diameter passing through the pivot point. In the absence of a gravitational force, show that the particle's motion in coordinate ϕ is the same as that of a simple pendulum in a uniform gravitational field. Explain why this result could have been expected.

7. Consider a double Atwood machine constructed as follows: A mass $4m$ is suspended from a string that passes over a massless pulley on frictionless bearings. The other end of this string supports a second similar pulley, over which passes a second string supporting a mass of $3m$ at one end and m at the other. Using two suitable generalized coordinates, set up the Lagrangian and use the Lagrange equations to find the acceleration of the mass $4m$ when the system is released. Explain why the top pulley rotates even though it carries equal weights on each side.

8. The figure shows a simple pendulum (mass m, length l) whose point of support P is attached to the edge of a wheel (center O, radius R) that is forced to rotate at a fixed angular velocity ω. At $t = 0$, the point P is level with O on the right. Write down the Lagrangian and find the equation of motion for the angle ϕ. [Hint: Be careful writing down the kinetic energy T. A safe way to get the velocity right is to write down the position of the bob at time t, and then differentiate.] Check that your answer makes sense in the special case that $\omega = 0$.