PHYS 620: Assignment 3

Due 9/20/2024

1. (a) Consider a rocket traveling in a straight line subject to an external force F^{ext} acting along the same line. Show that the equation of motion is

$$m\dot{v} = -\dot{m}v_{\text{ex}} + F^{\text{ext}}.$$

(b) Specialize to the case of a rocket taking off vertically (from rest) in a gravitational field g, so the equation of motion becomes

$$m\dot{v} = -\dot{m}v_{\text{ex}} - mg.$$

Assume that the rocket ejects mass at a constant rate, $\dot{m} = -k$ (where k is a positive constant), so that $m = m_0 - kt$. Solve the equation for v as a function of t, using separation of variables (that is, rewriting the equation so that all terms involving v are on the left and all terms involving t on the right). (c) Using the data: $m_0 = 2 \cdot 10^6$ kg, the final mass (after 2 minutes) of 10^6 kg, and the average exhaust speed v_{ex} of 3000 m/s, find the rocket's speed two minutes into flight, assuming that it travels vertically up during this period and that g doesn't change appreciably. Compare with the corresponding result if there were no gravity. (d) Describe what would happen to a rocket that was designed so that the first term on the right of equation written above was smaller than the initial value of the second.

2. Consider projectile motion in Earth gravitational field of an object of mass m subject to linear air resistance with drag coefficient b. It can be shown (do not show this) that ($\hat{\boldsymbol{y}}$ vector points vertically up)

$$x(t) = v_{x0}\tau\left(1 - e^{-t/\tau}\right)$$

$$y(t) = \tau(v_{y0} + v_{\text{term}})\left(1 - e^{-t/\tau}\right) - v_{\text{term}}t$$

where $\tau = m/b$, $v_{\text{term}} = mg/b = g\tau$, while v_{x0} and v_{y0} are initial velocities. As also shown, the (y, x) trajectory can be expressed as

$$y(x) = \frac{v_{y0} + v_{\text{term}}}{v_{x0}}x + v_{\text{term}}\tau\ln\left(1 - \frac{x}{v_{x0}\tau}\right).$$

   (a) In order to find the range of the motion: $y(x = R) = 0$, we expanded the logarithm up to cubic terms. When do you expect this approximation to be accurate?

   (b) After the expansion, we get a quadratic equation for R

$$\frac{2}{3v_{x0}\tau}R^2 + R - \frac{2v_{x0}v_{y0}}{g} = 0.$$

   Its solution involves a square root, which we expand as well, getting

$$R = \frac{2v_{x0}v_{y0}}{g}\left(1 - \frac{4v_{y0}}{3v_{\text{term}}}\right).$$

   This solution has an obvious problem that it can give $R < 0$. Discuss this finding to determine the conditions when the expansion is expected to be accurate.

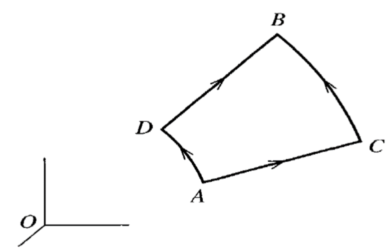
3. Which of the following forces is conservative? ***(a)*** $\boldsymbol{F} = k(x, 2y, 3z)$, where k is a constant; ***(b)*** $\boldsymbol{F} = k(y, x, 0)$; ***(c)*** $\boldsymbol{F} = k(-y, x, 0)$; ***(d)*** $\boldsymbol{F} = k(xy^2 + 3xz^2, x^2y + 2yz^2, 2y^2z + 3x^2z)$. For those which are conservative, find the corresponding potential energy $U(x, y, z)$ (you have to compute the appropriate line integral and clearly specify the curve (path) in three dimensional space used for the integration including the integration limits, i.e., the starting and ending coordinates of the path). Next, verify by direct differentiation that $\boldsymbol{F} = -\boldsymbol{\nabla} U$.

4. Show that

$$(a) \qquad \boldsymbol{\nabla} r^n = n r^{n-1} \hat{\boldsymbol{r}} \qquad (b) \qquad \boldsymbol{\nabla} f(r) = \hat{\boldsymbol{r}} \frac{df}{dr} \qquad (c) \qquad \boldsymbol{\nabla}^2 \ln r = \frac{1}{r^2}$$

5. A particle moves over the semicircle of radius 1 starting at the point (1,0) while subject to the force $\boldsymbol{F} = \mathrm{e}^y \hat{\boldsymbol{x}} + x\mathrm{e}^y \hat{\boldsymbol{y}}$. Calculate first the work performed using explicit line integration. Next, find a shortcut way for getting the answer.

6. Prove that a central and spherically symmetric force is conservative. Consider any two points A and B and two different paths ACB and ADB connecting them as shown in the figure. Path ACB goes radially out from A until it reaches the radius r_B of B, and then around a sphere (center O) to B. Path ADB goes around a sphere of radius r_A until it reaches the line OB, and then radially out to B. Explain clearly why the work done by a central, spherically symmetric force F is the same along both paths. This does not prove that the work is the same along any two paths from A to B. Complete the proof by showing that any path can be approximated by a series of paths moving radially in or out and paths of constant r.



7. Two bodies of masses m_1 and m_2 slide freely on a horizontal frictionless track and are connected by a spring with a force constant k.

   (a) Derive a single Newton equation describing the motion of this system in terms of the expansion of the spring ζ from the equilibrium length l.

   (a) Solve this equation and find the frequency of the oscillatory motion of the system.

8. The mass shown from above in the figure is resting on a frictionless horizontal table. Each of the two identical springs has force constant k and unstretched length l_0. At equilibrium, the mass rests at the origin, and the distances a are not necessarily equal to l_0. (That is, the springs may already be stretched or compressed.) ***(a)*** Find the expression for the potential energy of the mass when the mass moves to a position (x, y). Next, consider the case when x and y are small, and find an approximate potential energy including terms up the second power. ***(b)*** Based on the latter energy, find if the equilibrium at the origin is stable or not in dependence on the values of a and l_0. Next, validate your findings by simple physical arguments. *Hint:* You may need the following expansion $\sqrt{1+x} = 1 + x/2 - x^2/8 + \ldots$.