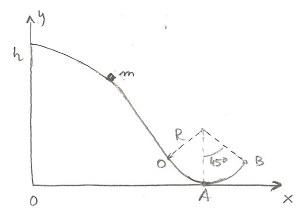
PHYS 620: Assignment 2

Due 9/13/2024

1. A block of mass m slides down a frictionless track shown in the figure. The track is of irregular shape except for the segment OAB which is circular with radius R. The arc AB forms the angle of 45°. The block is released from rest at $x = 0$ and at height h above the bottom of the loop (point A). The block is in the gravitational field of the Earth and there is no air resistance.
   (a) What is the force of the track on the block at point A?
   (b) What is the force of the track on the block at point B?
   (c) At what speed does the block leave the track (the track ends at point B)?
   (d) How far away from point A does the block land on level ground.
   (e) Sketch the potential energy $U(x)$ of the block. Indicate the total energy of the block on the sketch.



2. (a) Let $\boldsymbol{r}(t)$ be the position vector in two dimensions dependent on time t. Calculate the time derivative of the unit vector $\hat{\boldsymbol{r}}$ and show directly that this derivative is orthogonal to $\hat{\boldsymbol{r}}$ using explictly the components of these vectors. The vector $\hat{\boldsymbol{r}}$ and the unit vector in the direction of $\dot{\hat{\boldsymbol{r}}}$ can be chosen as the vectors defining the axes of the polar coordinate system. Draw a single picture showing both the Cartesian and polar coordinate systems with the unit vectors defining the axes of both systems explicitly marked.
   (b) Show using the general rules of differentiation that the time derivative of an arbitrary (multidimensional) vector of constant length is orthogonal to the vector.

3. Consider an object that is coasting horizontally (positive x direction) subject to a drag force $f = -bv - cv^2$, where v is the velocity and a and b are constants. The sum of all other forces acting on this object is zero. The initial velocity is v_0. Write down Newton's second law for the object and solve for v by separation of variables and taking into account the initial condition. Sketch the behavior of v as a function of time t. Find the time dependence for large t. Which force term is dominant when t is large?

4. A system consists of N masses m_α at positions $\boldsymbol{r}_\alpha$ relative to a fixed origin O. Let $\boldsymbol{r}'_\alpha$ denote the positions of the masses m_α relative to the center of mass (COM); that is, $\boldsymbol{r}'_\alpha = \boldsymbol{r}_\alpha - \boldsymbol{R}$, where $\boldsymbol{R}$ is the COM position.

(a) Make a sketch to illustrate this last equation.

(b) Prove the useful relation that $\sum m_\alpha \boldsymbol{r}'_\alpha = 0$. Can you explain why this relation is nearly obvious?

(c) Use this result to show that the rate of change of the angular momentum about COM is equal to the total external torque about the COM:

$$\dot{\boldsymbol{L}}(\text{COM}) = \boldsymbol{\Gamma}^{\text{ext}}(\text{COM}).$$

Note that this result is nontrivial since the COM may be accelerating, so it is not neccessarily a fixed point in any inertial frame.

5. A bead is sliding on a rigid wire curved in a three-dimensional space. The position of the bead can be specified by its distance s measured along the wire from the origin of the wire. The bead is subject to an external conservative force $\boldsymbol{F}_{\text{ext}}(\boldsymbol{r})$.

   (a) Prove that the bead's speed $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ is equal to the magnitude of $\dot{s} = ds/dt$.

   (b) Prove that $m\ddot{s} = F_{\text{tang}}$, where F_{tang} is the component of the net force acting on the particle that is tangential to the wire.

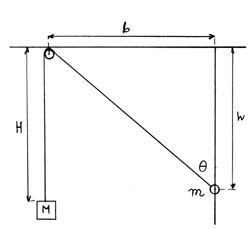
   (c) Prove that $F_{\text{tang}} = -dU/ds$, where U is the potential of the external force: $\boldsymbol{F}_{\text{ext}}(\boldsymbol{r}) = -\boldsymbol{\nabla} U(\boldsymbol{r})$.

6. A metal ball of mass m with a hole through it is threaded on a frictionless vertical rod. A massless string of length l is attached to the ball, runs over a massless, frictionless pulley, and supports a block of mass M, as shown in the figure. The distance between the rod and the pulley is b. Note that the positions of the two masses can be specified by a single angle θ.

   (a) Find the potential energy as a function of θ, $U = U(\theta)$. Assume that the pulley and the ball have negligible size.

   (b) Using U, find whether the system has an equilibrium position. Consider the cases $m > M$, $m = M$, and $m < M$. Discuss the dependence on l and stability of any equilibrium position.

   (c) Interpret the case of equilibrium position (if any) by considering the forces acting on the ball and block.



7. An unusual pendulum is made by attaching a string to a horizontal cylinder of radius R, wrapping the string several times around the cylinder, and then tying a mass m to the loose end. The cylinder is fixed in space. In equilibrium, the mass hangs a distance l_0 vertically below the edge of the cylinder (i.e., l_0 is the length of the free segment of the string). Find the potential energy U if the pendulum has swung to an angle ϕ from the vertical. Show that for small angles, it can be written in the Hooke's law form $U = \frac{1}{2}k\phi^2$.

8. A grandfather's clock has a pendulum of length l with a bob of mass m and is in a constant gravitational field of acceleration g. To counteract damping, a mechanism gives an almost instantaneous "kick" to the pendulum at the end of each cycle, restoring the amplitude to a constant value θ_m. The mechanism uses the gravitational energy of a weight of mass M which falls a distance h in time d. Since the amplitude is small, one can assume that the pendulum behaves like a standard damped harmonic oscillator between the kicks. Assume that the only energy losses in the system are due to the damping of the pendulum. In other words, approximate the system as a damped harmonic oscillator whose amplitude is instantaneously restored to the inital value at the end of each cycle. Our goal is to find the damping force.

   (a) Write down the Newton equation for the pendulum (in the limit of small oscillations) in terms of the angular deviation θ from the equilibrium and then the solution to this equation (no need to derive these equations here).

   (b) Find the gravitational energy loss of the weight during the time of one cycle τ (time between the kicks).

   (c) Define, in general terms, the mechanical energy dissipated by the pendulum in one cycle. Note that to calculate this quantity, one needs to use only the potential energy.

   (d) Now find the detailed expression for the potential energy of the pendulum $U(t)$ in terms of m, g, l, and $\theta(t)$.

   (e) Find first an expression for the positions of maxima of a damped harmonic oscillator and then calculate the energy dissipated in one cycle. To simplify the expression, show that although the cosine function appearing in the solution is not equal to 1 at the maxima of $\theta(t)$, its value is the same at all the maxima.

   (f) Use the given quantity θ_m to eliminate the arbitrary constants in the expression for the dissipated energy.

   (h) Compare the loss of the gravitational energy of the weight and the dissipated energy to find an expression for the damping constant of the pendulum in terms of known quantities. Then find an approximate expression for small damping.

9. A particle of mass m slides without friction on a horizontal table. The particle is attached to one end of a massless spring of equilibrium length a and spring constant k. The other end of the spring is attached to a point on the table, such that the spring can rotate around this point without friction.

(a) What is the net force acting on the particle? Note that the gravity force is balanced by the reaction of the table and there is no friction.

(b) Find the expression for the kinetic energy T, the potential energy U, and the magnitude of the angular momentum $|\boldsymbol{l}|$ of the particle in cylindrical coordinates. Then express T in terms of $|\boldsymbol{l}|$.

(c) Define the effective potential energy U_{eff} as the sum of U and the part of T depending on $|\boldsymbol{l}|$. Sketch the U and U_{eff} potential energy functions.

(d) Find the angular velocity of the rotational motion of the particle ω_0 such that the particle moves exactly on a circular orbit of radius r_0. Note that ω_0 is not equal to $\sqrt{k/m}$.

(e) Relate ω_0 to U_{eff}. Interpret this result physically.

(f) Argue qualitatively that for angular velocities ω larger than ω_0, the particle will perform some kind of oscillatory motion in the radial coordinate.

(g) Show that if ω is close to ω_0, the particle will oscillate around the orbit of radius r_0 and the motion in the radial coordinate will be that of a simple harmonic oscillator. Find the frequency of such small oscillations. *Hint:* Show that U_{eff} can be approximated by a harmonic oscillator potential. Then find the solution from the total energy expression derived in points (b) and (c).

Express all you answers in term of m, k, a, and r_0.

10. Two carts of masses m_1 and m_2 move on a linear, horizonal, frictionless track. The carts are connected by a harmonic spring with the spring constant k and the equilibrium length l. Find the position of each cart as the function of time.

(a) To separate the center of mass (COM) and the relative motions, write down the formulas relating the x_1 and x_2 coordinates of the carts to the COM coordinate x_{CM} and the relative coordinate $x = x_2 - x_1$.

(b) Write down Newtons equations for x_1 and x_2, transform them into Newton's equations for x_{CM} and x, and solve the latter equations.

(c) Assume the following values of parameters: $m_1 = m_2 = 0.1$ kg, $k = 1$ kg/s^2, and $l = 0.1$ m, as well as the following initial conditions: $x_1(0) = 0$, $x_2(0) = 0.15$ m, $v_1(0) = 0.1$ m/s, and $v_2(0) = 0.1$ m/s. Plot on one graph $x_1(t)$ and $x_2(t)$, on another graph $\dot{x}_1(t)$ and $\dot{x}_2(t)$, and on one more graph $\ddot{x}_1(t)$ and $\ddot{x}_2(t)$. Discuss these results from the point of view of Newton's laws.

(d) Same as in point (c) for $m_2 = 0.3$ kg. What new critical information such experiment would provide for the discovery of Newton's laws.