PHYS 620: Assignment 1

Due 9/6/24

1. (a) Assume that you can measure only the position of a particle and time. Formulate the Newton's law for two isolated bodies using only these quantites.

   (b) Use the results of (a) to define mass. With this definition, write the Newton's law as one concise equation.

   (c) Use the results of (a) and (b) to define force and show how the three traditional Newton's laws result.

   (d) Using only the laws and definitions from (a)-(c), derive the conservation of momentum theorem for two particles.

2. A particle of mass m moves horizontally in a medium under the influence of a retarding force $-bv - dv^3$. There are no other forces acting on the particle. The initial velocity is $v(0) = v_0$ and the initial position is $x(0) = 0$. Solve Newton's equation of motion to find the dependence of the position of the particle on time, $x(t)$. Then show that no matter how large v_0 is, the particle will never move beyond $x_{\max} = m\pi/(2\sqrt{bd})$. *Hints:* Use 'partial fractions in the first integration. In the second integration, substitute the exponential by a simple variable, $\int dx/\sqrt{1-x^2} = \arcsin x$.

3. Consider a pulley of radius R and moment of inertia I with two blocks of masses m_1 and m_2 suspended from it on a massless inextensible string that passes over the pulley. The masses hang vertically in Earth's gravitational field with a gravity acceleration g. The pulley rotates without friction on its axis. The string moves with the pulley without slipping.

   (a) Write down the total energy E of the system (two masses and the pulley) exclusively in terms of the coordinate x_1 of the mass m_1 and possibly in terms of the derivatives of x_1. *Hint:* Do not forget about the kinetic energy of the pulley equal to $I\omega^2/2$, where ω is the angular velocity of the rotation.

   (b) Find the equation of motion for x_1 by differentiating the energy with respect to time.

   (c) To check the answer from point (b), write down Newton's equations for the system of two masses and the pulley and eliminate the two unknown tensions from the three resulting equations.

4. Consider a damped harmonic oscillator of mass m, harmonic constant k, and damping constant b for the case with $\beta < \omega_0$, where $\beta = b/2m$ and $\omega_0 = \sqrt{k/m}$. There is a little difficulty defining the "period" since the motion is not periodic. However, a

definition that makes sense is that such a period, which will be denoted by τ_1, is the time between successive maxima of $x(t)$. Make a sketch of $x(t)$ against t and indicate this definition of the period on your graph. Show first that the maxima of $x(t)$ are at different t than the maxima of $\cos(\omega_1 t - \delta)$, see hints for notation. Then find τ_1. *Hints:* For a damped oscillator described by Newton's equation $m\ddot{x} = -kx - b\dot{x}$ the solution is $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$, where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$, while A and δ are adjustable constants.

5. A simple pendulum consists of a mass m suspended from a fixed point in Earth's gravitational field (acceleration g) by a weightless rigid rod of length l. The pendulum moves in a viscous medium with retarding force $\boldsymbol{F}_{\text{damp}} = -b\boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity. Use notation $\omega_0 = \sqrt{g/l}$ and $\beta = b/2m$.

   (a) Write down the equation of motion in two dimensions, derive from it the equation of motion in one dimension, and find an approximation to it in the limit of small oscillations (*Hints:* In polar coordinates $\ddot{\boldsymbol{\rho}} = (\ddot{\rho} - \rho\dot{\theta}^2)\hat{\boldsymbol{\rho}} + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\hat{\boldsymbol{\theta}}$; $\sin x \approx x$ for small x).

   (b) For the case $\boldsymbol{F}_{\text{damp}} = -2m\sqrt{g/l}\,\boldsymbol{v}$, solve the approximate equation of motion. How many solutions do you get? *Hint:* Try an exponential guess.

   (c) If you get only one solution, consider the case $\beta < \omega_0$, solve this equation, and deduce the other solution from the limit behaviour $\beta \to \omega_0$.

6. Imagine two concentric cylinders, centered on the vertical $\hat{\boldsymbol{z}}$ axis in Earth gravitational field, with radii $R \pm \epsilon$, where ϵ is very small. A small frictionless puck of thickness 2ϵ and mass m is inserted between the two cylinders, so that it can be considered a point mass that can move freely at a fixed distance from axis $\hat{\boldsymbol{z}}$. Use cylindrical coordinates to solve Newton's equations of motion for the puck. Next, describe this motion verbally. The second derivative of vector $\boldsymbol{r}$ in cylindrical coordinates is

$$\ddot{\boldsymbol{r}} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\boldsymbol{z}}$$

7. Consider a uniform solid disk of mass M and radius R, rolling without slipping down an incline which is at an angle γ to the horizontal in Earth's gravitational field. Find the linear acceleration $\dot{v}$ of the center of mass of the disk along the incline following the steps listed below.

   (a) Show first that $v = R\omega$, where ω is the angular velocity of the rotation of the disk around its center of mass.

   (b) Denote by P the point of the instantaneous contact between the disk and the incline. Show that the angular velocity of the rotation of the disk around P is also ω.

(c) Use the Newton's law for rotational motion about P to find $\dot{v}$. Note that we pick one particular point P, fixed with respect to the incline, so that the coordinate system with the orgin at P is inertial. The moment of inertia for the rotation of a disk around a point on its circumference is $\frac{3}{2}MR^2$.

8. (a) Using only the laws and definitions from problem 1, formulate the Lagrange equations in Cartesian coordinates for a single unconstrained particle of mass m in an external force field $\boldsymbol{F}$. The Lagrangian is defined as $\mathcal{L} = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - U(\boldsymbol{r})$, where the potential energy U satisfies $\boldsymbol{F} = -\boldsymbol{\nabla}U(\boldsymbol{r})$.

   (b) Use the Euler-Lagrange relations to extend the Lagrange equations to arbitrary coordinate systems.

9. Show that for a particle moving on a fixed surface in a conservative force field the functional $S[\boldsymbol{R}] = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{R}, \dot{\boldsymbol{R}}, t)dt$, where the $\mathcal{L}$ is the difference of the kinetic and potential energies of the particle, $\mathcal{L} = T - U$, is minimized by $\boldsymbol{r}$, the solution of the Newton's equations. *Hints:* Write $\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{\epsilon}$ and develop an expression for the contribution to S linear in $\boldsymbol{\epsilon}$. This requires Taylor's expansion of the potential energy U and an integration by parts. Next relate $\boldsymbol{\nabla}U$ to the forces present in this problem.

10. A double pendulum is attached to a cart of mass $2m$ which moves without friction on a horizontal surface. Each pendulum has length b, the mass of the bob m, and the rod is massless. Find the equations of motion for the system. Do not solve these equations.