Review of Classical Mechanics in Fundamentals of Physics (PHYS 207/227)

I. TRICKS OF THE TRADE

- Always check the dimensions (units) of expression on both sides of you equations. For examples, one can immediately see that the equation "acceleration = force" cannot be correct since acceleration and force are expressed in different units (m/s^2 vs. newtons).
- Always try to solve problem using symbols a long as possible and only in the final stage substitute numbers.
- In solving problems, draw figures illustrating the system. You may start from a "picture" of the system, but later move to a more abstract diagram with bodies shown as point masses. The latter drawing is called a "free-body" diagram (it may include more than one body). All the forces acting on each body should be marked.
- Always mark directions of axes on you figures.
- Make sure that none of your expressions have a vector on one-hand side and a scalar on the other-hand side (vector never equals scalar).
- Vectors such a forces should be always added, not subtracted, e.g.,

$$\boldsymbol{F}^{\text{net}} = \boldsymbol{F}^A + \boldsymbol{F}^B + \boldsymbol{F}^C.$$

  Subtraction of vectors is mathematically well defined, but when we calculate net forces, the signs should be taken care by individual vectors.
- Similarly, components of vectors should be summed with plus signs

$$F_x^{\text{net}} = F_x^A + F_x^B + F_x^C.$$

- In one-dimensional cases, it is often convenient to replace components by magnitudes of the vectors. Magnitudes have to be sometimes added and sometimes subtracted, so it is safer to use components.
- Always write Newton's equations separately for each body in the system even if we know that the bodies move together. Attempts to write as single Newton equation for a set of bodies are prone to errors.
- Make sure that the force in your Newton's equation is the net force, i.e., the sum of all forces acting on a given body. Mark all such forces on a "free-body" diagram.

- If for some reasons you need a single Newton equation for a set of bodies moving together, the best way to get it is by adding Newton's equations for individual bodies.

- Remember Riemann: most quantities in this course are defined first on a discrete set of objects and then in the limit of the objects becoming infinitely small we get integral expressions (examples: work and moment of inertia).

- In general it is "safer" to use momentum conservation than energy conservation since the latter is true only for conservative forces. The momentum conservation, on the other hand, requires that no external forces (or torques) act on the system, which always has to be checked. In some cases, like elastic collisions, we often have to use both theorems.

- When considering rigid bodies, we use two types of distances: distance from the center of a coordinate system, denoted by $r = |\boldsymbol{r}|$, and the distance from an axis in space, denoted by $\mathscr{r}$.

II. KINEMATICS OF LINEAR MOTION

A. Vectors

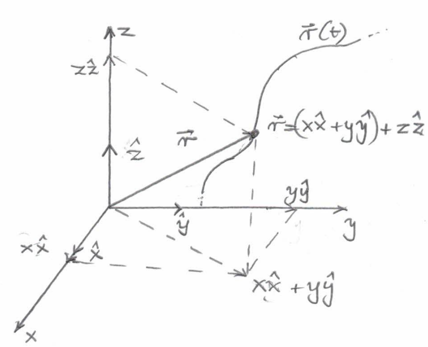
We will use all the time quantities which are three-dimensional vectors, i.e., have three components: $\boldsymbol{A} = (A_x, A_y, A_z)$. This concept should be well known from math courses. A set of all vectors forms a vector space, with properties such addition of vectors, multiplication of vectors by a number, and scalar product of two vectors: $\boldsymbol{A} \cdot \boldsymbol{B}$, which is a number.

B. Position vector

We will denote the position of a point in space by a vector $\boldsymbol{r}$. If we define an arbitrary Cartesian coordinate system, this vector can be described by a set of three numbers:

$$\boldsymbol{r} = [x, y, z] = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}, \tag{1}$$

where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ are unit vectors along the axes of the coordinate system. These vectors are exactly the same as the unit vectors denoted by $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$, respectively, used in some textbooks. The quantities x, y, and z are called coordinates of vector $\boldsymbol{r}$. This concept is illustrated in the figure. Using the vector addition rules, we first see that the sum of vectors $x\hat{\boldsymbol{x}}$ and $y\hat{\boldsymbol{y}}$ is a vector drawn



in the xy plane. The sum of this vector and of the vector $z\hat{\boldsymbol{z}}$ is the vector $\boldsymbol{r}$. We use $\boldsymbol{r}$ to denote a position of a particle in space. By particle we will understand a small body. We often call such body a "point-like" particle.

C. Vector fields

When a particle is in point $\boldsymbol{r}$ and a force $\boldsymbol{F}$ acts on this particle, it is convenient to use notations $\boldsymbol{F}(\boldsymbol{r})$ to indicate this relation. Such six-dimensional objects are called vector fields. Thus, at each point $\boldsymbol{r}$ there is a separate vector space formed by all possible vectors $\boldsymbol{F}(\boldsymbol{r})$. The concept of vector field is also convenient to describe the forces in space due, for example, to the gravity attraction of a body such as the Earth.

D. Trajectory, velocity, acceleration

If the particle is moving in the coordinate system, $\boldsymbol{r} = \boldsymbol{r}(t)$. Thus, each component of $\boldsymbol{r}$ is a single-variable function, e.g., $x = x(t)$. The path in three-dimensional space that the particle travels is called trajectory:

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{x}} + y(t)\hat{\boldsymbol{y}} + z(t0\hat{\boldsymbol{z}}.$$

Note that the unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ do not change in time.

In physics we are interested in the rate of change of $\boldsymbol{r}(t)$, i.e., in the derivative of $\boldsymbol{r}(t)$ with respect to time. This derivative is called velocity and denoted by $\boldsymbol{v}(t)$. How does one compute derivative of a vector? We can reduce this derivative to the familiar derivatives of $x(t)$, $y(t)$, and $z(t)$:

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}(t)}{dt} = \frac{d(x(t)\hat{\boldsymbol{x}})}{dt} + \frac{d(y(t)\hat{\boldsymbol{y}})}{dt} + \frac{d(z(t)\hat{\boldsymbol{z}})}{dt} = \hat{\boldsymbol{x}}\frac{dx(t)}{dt} + \hat{\boldsymbol{y}}\frac{dy(t)}{dt} + \hat{\boldsymbol{z}}\frac{dx(t)}{dt},$$

where we first used the fact that the derivative of a sum of functions is the sum of derivatives of these functions and than the fact that the unit vectors are constants of time.

In a similar way, we can consider the rate of change of $\boldsymbol{v}(t)$, called acceleration and denoted by $\boldsymbol{a}(t)$

$$\boldsymbol{a}(t) = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2}.$$

Can we reverse this procedure and get $\boldsymbol{v}(t)$ from $\boldsymbol{a}(t)$ and then $\boldsymbol{r}(t)$ from $\boldsymbol{v}(t)$. We know from calculus that it is possible by integration. Again, the problem reduces to a familiar one-dimensional case:

$$\int_{t_1}^{t_2} \boldsymbol{v}(t)\,dt = \int_{t_1}^{t_2} \frac{d\boldsymbol{r}(t)}{dt}\,dt = \hat{\boldsymbol{x}} \int_{t_1}^{t_2} \frac{dx(t)}{dt}\,dt + \hat{\boldsymbol{y}} \int_{t_1}^{t_2} \frac{dy(t)}{dt}\,dt + \hat{\boldsymbol{z}} \int_{t_1}^{t_2} \frac{dz(t)}{dt}\,dt =$$
$$= [\hat{\boldsymbol{x}}x(t) + \hat{\boldsymbol{y}}y(t) + \hat{\boldsymbol{z}}z(t)]_{t_1}^{t_2} = \boldsymbol{r}(t_2) - \boldsymbol{r}(t_1), \tag{2}$$

where t_1 and $_2$ are two instants of time, we used the linearity property of integral, constancy of unit vectors and the fact that the integral of the derivative of a function is equal to this function. In a similar way:

$$\boldsymbol{v}(t_2) - \boldsymbol{v}(t_1) = \int_{t_1}^{t_2} \boldsymbol{a}(t)\,dt.$$

E. Example: constant acceleration, $\boldsymbol{a} = $ const.

The equations written above simplify if $\boldsymbol{a}(t) = $ constant (while this is an important example, most problems in 419 will not consider this case). It will be convenient to rename some variables: $t_1 \to t_0$, $t_2 \to t$, $t \to t'$. We can write then

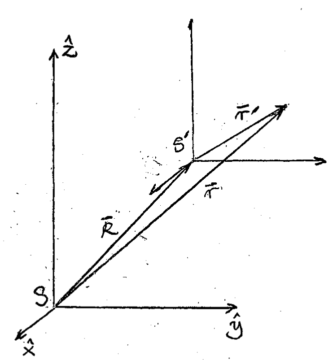
$$\boldsymbol{v}(t) = \boldsymbol{v}(t_0) + \int_{t_0}^{t} \boldsymbol{a}\,dt' = \boldsymbol{v}(t_0) + \boldsymbol{a} \int_{t_0}^{t} dt' = \boldsymbol{v}(t_0) + \boldsymbol{a}\,(t - t_0),$$

where we could pull $\boldsymbol{a}$ out of integral since it is a constant of time. We now integrate this expression:

$$\boldsymbol{r}(t) = \boldsymbol{r}(t_0) + \int_{t_0}^{t} \boldsymbol{v}(t')\,dt' = \boldsymbol{r}(t_0) + \int_{t_0}^{t} [\boldsymbol{v}(t_0) + \boldsymbol{a}\,(t' - t_0)]\,dt'$$
$$= \boldsymbol{r}(t_0) + \boldsymbol{v}(t_0) \int_{t_0}^{t} dt' + \boldsymbol{a} \int_{t_0}^{t} (t' - t_0)\,dt' = \boldsymbol{r}(t_0) + \boldsymbol{v}(t_0)(t - t_0) + \frac{1}{2}\boldsymbol{a}\,(t - t_0)^2, \tag{3}$$

F. Addition of velocities

Consider a system S which is at rest and a system S' moving with respect to system S with a constant velocity $\boldsymbol{V}$. The axes of the two systems are parallel and the origin of S' is at $\boldsymbol{R}$. The position of a particle in S' (S) is denoted by $\boldsymbol{r}'$ ($\boldsymbol{r}$), see the figure. The velocities are



$$\boldsymbol{V} = \frac{d\boldsymbol{R}}{dt}; \qquad \boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}; \qquad \boldsymbol{v}' = \frac{d\boldsymbol{r}'}{dt}.$$

Since

$$\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{r}' \quad \Rightarrow \quad \boldsymbol{v} = \boldsymbol{V} + \boldsymbol{v}', \tag{4}$$

where we simply differentiated the first equation. What we got is the velocity addition theorem. It is true within Galilean relativity valid in classical mechanics and obviously has to be modified in relativistic mechanics where no object can move with a speed larger than the speed of light.

III. KINEMATICS OF CIRCULAR MOTION

A. Angular displacement, velocity, acceleration

Consider a point moving on a circle of radius r. We can describe its position using the angle θ, see the figure. The measure of angle θ in radians is defined as

$$\theta = \frac{s}{r},$$

where s is the length of the arc spanned by θ. Thus, $\theta(t)$ is the angular displacement of the point, while $s(t)$ the linear (tangential) displacement. The angular velocity and acceleration are then defined as

$$\omega(t) = \frac{d\theta(t)}{dt}, \qquad \alpha(t) = \frac{d\omega(t)}{dt} = \frac{d^2\theta(t)}{dt^2}.$$

B. Relation to linear displacement s and radius r

We also have

$$\omega(t) = \frac{d\theta(t)}{dt} = \frac{d(s(t)/r)}{dt} = \frac{v}{r} \qquad \text{or} \quad v = \omega r, \tag{5}$$

and similarly

$$\alpha(t) = \frac{a_t}{r} \quad \text{or} \quad a_t = \alpha\, r \tag{6}$$

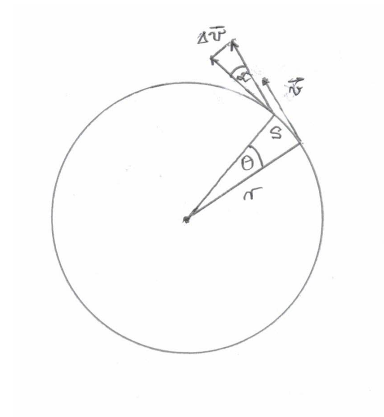
where $v = \frac{ds}{dt}$ and $a_t = \frac{dv}{dt}$ is the tangential acceleration.

C. Centripetal acceleration

Consider motion of a point on the circle with constant magnitude of velocity: $v = |\boldsymbol{v}|$. Although v is constant, $\boldsymbol{v}$ is not since its direction changes. Define $\Delta\boldsymbol{v} = \boldsymbol{v}(t_2) - \boldsymbol{v}(t_1)$. The rate of change of $\boldsymbol{v}$ is called centripetal acceleration

$$\boldsymbol{a}_c = \frac{d\boldsymbol{v}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{v}}{\Delta t}. \tag{7}$$

The quantity on the right-hand side is illustrated in the figure. The triangle formed by $\boldsymbol{v}(t_2)$ and $\boldsymbol{v}(t_1)$ shifted in such a way that the origins of the two vectors overlap is isosceles and the angle $\alpha = \theta$ (since the arms of one angle are perpendicular to the arms of the other angle). If we divide this triangle into two right triangles, we have the relation



$$\frac{|\Delta \boldsymbol{v}|/2}{v} = \sin(\theta/2).$$

When $\theta \to 0$, $\sin(\theta/2) = \theta/2$. Thus, we have then

$$|\Delta \boldsymbol{v}| = \theta v$$

and

$$\boldsymbol{a}_c = \frac{d(\theta v)}{dt} = v\omega = \frac{v^2}{r}(-\hat{\boldsymbol{r}}),$$

where $\boldsymbol{r}$ is the vector from the center of the circle to the body, so that $\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}$. The fact that $\boldsymbol{a}_c$ is radial comes from the observation that when $\theta \to 0$, the opposite side in the right triangle considered becomes perpendicular to the hypotenuse.

IV. MEASUREMENTS

While the developments in Secs. II and III can be viewed as mathematical statements, in physics one has to include one more elements: measurement. The world measurement means determination of the size or magnitude of something. This is achieved by comparing a given quantity to a standard quantity of the same nature. This standard quantity is called the unit of measurement. For example, for length, i.e., for the quantities x, y, and z in Secs. II the unit is meter (m). Initially, it was just a bar deposited at Bureau international des poids et mesures (BIPM) near Paris. For time, the unit is second (s) and it was initially defined as a fraction of one day. Currently, second is defined via the frequency of light needed to make transition between two two particular states of the cesium (Cs) atom. Similarly, meter is defined as a fraction of the distance traveled by light in one second.

V. NEWTON'S LAW(S), MASS, FORCE

See Lecture Notes "Newton's Law" for a complete discussion of this subject. In all texts at the fundamentals of physics level, Newton's (2nd) law is introduced by stating that acceleration of a body is proportional to the force acting on it and inversely proportional to mass of the body: $\mathbf{a} = \mathbf{F}/m$. The problem with this definition is that quantities on the right-hand side of this equation are only vaguely defined by invoking everyday experiences.

A rigorous way of formulating Newton's law uses on accelerations, quantities which are clearly defined, see Secs. II and IV. Mass and then force are then *defined* using Newton's law.

Newton's law is a statement based on experimental observations and it cannot be derived from some other statements. This is in general the meaning of the word "law" in physics. In contrast, the word "theorem" denotes a statement that can be derived using logical reasoning from some set of axioms (assumptions) and earlier theorems. Almost all of classical mechanics that we will deal with can be derived from Newton's law.

A. Single Newton's law

We formulate Newton's law based on an experiment that can be watched at `http://www.-youtube.com/watch?v=amfw2nABke4`. The ideal system for this experiment should consist of two completely *isolated* bodies. The bodies interact only with each other and there is no external "agent" which would affect the motions of these bodies. The best approximation for an isolated system are two bodies in space far from any stars or planets. The carts in the video are an approximation to such a system. They move on an air track with negligible friction. The external agent does start the motion, but later does not influence it. We do not know any details concerning the flexible wire connecting the cars.

Based on measurements of the positions of the carts in the video, one can draw the graph that can be found in Lecture Notes on "Newton's Law". At the bottom part of this graph, one can see that the accelerations of the two carts are equal in magnitude and of opposite sign. If we replaced one of the carts by some other one, the ratio of the magnitudes would still be constant in time, but in general not equal to one anymore. These observations can be generalized to the following statement of Newton;'s law:

For two isolated bodies, their accelerations vectors are parallel, opposite, and

$$\frac{|\boldsymbol{a}_2|}{|\boldsymbol{a}_1|} = k_{21}$$

where k_{21} is a constant. An equivalent formulation is:

$$\boldsymbol{a}_2 = -k_{21}\boldsymbol{a}_1, \quad k_{21} > 0.$$

B. Mass

Mass is defined by writing the constant in Newton's law as

$$k_{21} = \frac{m_1}{m_2}$$

which allows us to write Newton's equation as

$$m_2\boldsymbol{a}_2 = -m_1\boldsymbol{a}_1.$$

If one chooses a body and sets its mass to one (unit mass), masses of all other bodies become uniquely defined. The unit of mass is kilogram (kg), which was initially defined as the mass of a platinum-iridium cylinder kept at BIPM. Later it was redefined based on the mass of carbon isotope ^{1}2C. The current definition is less straightforward and it used the Planck constant (a constant that relates photon frequency to photon energy), fixed at some value, together with definitions of meter and second.

The fact that we got $k_{21} = 1$ in the experiment watched is due to equal masses of both carts. It would be good to have another version of this experiment with unequal masses.

C. Force

Force is simply defined as

$$\boldsymbol{F} = m\boldsymbol{a}.$$

We now see that mass is a measure of inertia of a body, i.e., how hard it is to accelerate it applying a given force. Force is measured in newtons (N), 1 N = $\mathrm{kg\,m/s^2}$.

D. Traditional Newton's laws

What we introduced above, is a single Newton's law, followed by the definitions of mass and force. The three "traditional" Newton's laws follow from this law.

- **1st NL:**
  From the definition of force
  $$\boldsymbol{F} = 0 \Leftrightarrow \boldsymbol{a} = 0.$$
  We say that if no force is acting on a body, it remains at rest or moves with a constant velocity.

- **2nd NL:**
  This is just our definition of force
  $$\boldsymbol{F} = m\boldsymbol{a}$$

- **3rd NL:**
  From Newton's law and the definitions of mass and force
  $$\boldsymbol{F}_1 = m_1\boldsymbol{a}_1 = -m_2\boldsymbol{a}_2 = -\boldsymbol{F}_2 \quad \Rightarrow \quad \boldsymbol{F}_1 = -\boldsymbol{F}_2.$$
  We say that if body 1 acts on body 2 with force $\boldsymbol{F}_2$, body 2 acts on one with force $-\boldsymbol{F}_2$.

E. Inertial coordinate system

We define an inertial coordinate system (frame) as such system where Newton's law holds. If a system S is inertial, a system S' moving with a constant velocity with respect to S is also inertial (see the figure in Sec. II F). This theorem comes directly from Eq. (4): differentiating the velocity addition theorem, we get $\boldsymbol{a} = \boldsymbol{a}'$ (since $\boldsymbol{V}$ is constant), thus, the accelerations are the same in S and in S'. The best approximation to an inertial system is a system fixed on stars. For most practical purposes, a system fixed on Earth is a good enough approximation, although there are several experiments that show the noninertial character of this frame.

VI. GRAVITY

A. Newton's law of gravity

The experimentally determined Newton's gravity law states that two point-like bodies of masses m_1 and m_2 attract each other with the force

$$|\boldsymbol{F}_g| = G\frac{m_1 m_2}{r^2}$$

where r is the distance between the bodies. More precisely, the force on body 1 due to body 2 is

$$\boldsymbol{F}_{g,1(2)} \equiv \boldsymbol{F}_{g,12} = G\frac{m_1 m_2}{r^2}(-\hat{\boldsymbol{r}})$$

where $\boldsymbol{r} \equiv \boldsymbol{r}_{12} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ is the vector pointing from body 2 to body 1. The fact that the masses in Newton's law (inertial) and in Newton's law of gravity (gravitational) are the same is an experimental observation. The constant G, called gravitational constant, is obtained from measurements and is equal to $0.67\,10^{11}$ $\mathrm{m}^3/(\mathrm{kg\,s}^2)$.

B. Solid sphere interacting with a point mass

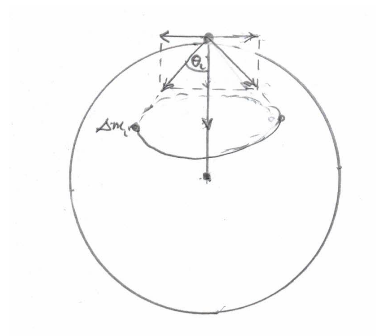
Newton's law of gravity can be shown to be valid for a solid sphere of uniform density or a spherical shell interacting with a point mass. The force on point mass outside the sphere or the shell is then given by the same equation as written above except that now the distance r is between the center of mass of the sphere and the point-like body. For a point mass inside the shell, the force of gravity is zero. The proof of this theorem will be given in 419.

C. Gravitational force near the surface of the Earth

Using the theorem formulated above, we can find immediately that the force acting on a point-like body on the Earth surface is

$$|\boldsymbol{F}_g| = G\frac{M_e m}{R_e^2} = mg \quad \text{where} \quad g = G\frac{M_e}{R_e^2}.$$

In the equation above, M_e and R_e are Earth's mass and radius. Thus, the acceleration $g = 9.8$ m/s^2 can be computed from known quantities. In this approach, free-fall equation is a theorem resulting from Newton's laws.



To avoid the use of unproved theorem of Sec. VI B, we can use the following reasoning. Divide the sphere representing the Earth into small point-like masses Δm_i. We can now apply Newton's gravity low in the form of Sec. VI A. Due to spherical symmetry, for each mass Δm_i, there is another mass that cancels the tangential component of the force of gravity, see the figure. This means that the sum of all forces is directed towards the center of the Earth. We can now sum all such components from the masses Δm_i

$$|\boldsymbol{F}| = \left| mG\sum_i \frac{\Delta m_i}{{r_i}^2}(-\hat{\boldsymbol{r}}_i) \right| = mG\sum_i \frac{\Delta m_i \cos\theta_i}{{r_i}^2} = mg$$

where $\boldsymbol{r}_i$ the distance from Δm_i to m and θ_i is the angle marked on the figure. We do not compute the sum, but it obviously gave a constant value. We denoted the sum of all constants by g, but we do not know its value. However, we can just measure g.

D. Kepler's laws

The main success of Newton's theory was to prove Kepler's laws on motions of planets. Thus, Kepler laws became theorems (but one continues to call the laws). The Ptolemaic geocentric model from second century was able to predict these motions very well, but was very complicated. The Copernican revolution introduced the heliocentric model in 1543. Interestingly, this model was less accurate than the Ptolemaic one, but was much simpler. The accuracy issue was resolved by Kepler in years 1609-1619 by the introduction of elliptic orbits. With Newton's work, Kepler's laws became theorems.

(1) Planets move on elliptical orbits with the Sun in ellipses' foci

This law is only formulated in fundamentals of physics courses, we will prove it in 419.

(2) The vector from the sun to the planet sweeps equal areas in equal times

This law is easy to prove and we will do it in this review after covering the concept of angular momentum.

(3) The squares of periods of motions are proportional to the third power of the distance from Sun

This low is proved in fundamentals of physics courses only for circular orbits. We found that circular motion results in the centripetal acceleration, Eq. (7). This acceleration in case of the motion of a planet around Sun is provided by the force of gravity:

$$F_g = G\frac{Mm}{r^2} = ma_c = \frac{mv^2}{r},$$

where M is the mass of the Sun, m is the mass of the planet, v is the linear velocity of the planet, and r is the distance between these two bodies. The relation between v and the period of the motion T is

$$v = \frac{2\pi r}{T}.$$

Combining these equations, we get

$$v = \sqrt{\frac{GM}{r}} = \frac{2\pi r}{T} \quad \Rightarrow \quad T^2 = \frac{4\pi^2}{GM} r^3.$$

VII. WEIGHT

- **Weight of a body:**
  Force needed to prevent the body from falling in a gravitational field. Thus, the body is at rest (but can also move with a constant velocity). This force can be a normal force from the surface on which the object is resting or a tension force from a string on which the object hangs. Note that while mass is an intrinsic property of a body, weight depends on where the body is and is different on the surface of the Earth and on the surface of the Moon. Example: elevator at rest or moving with a constant velocity. The normal force from the floor is the weight. Note that the net force acting on a person in the elevator is zero since the force of gravity is cancelled by the normal force.

- **Apparent weight (or perceived weight):**
  As above for accelerating objects (includes forces needed to accelerate a body). Example: elevator accelerating up. In addition to the normal force, the accelerating force act on the body. Our perceived weight is larger now than our normal weight. If we stand on a scale, the scale will show this perceived weight. In contrast, when the

elevator accelerates down, the scale will show a reduced weight. The limit case is the elevator in free fall, with zero perceived weight (we feel weightless).

VIII. FRICTION

The force of friction opposes the motion and its magnitude is given by

$$F_f = \mu F_n \qquad \mu = \mu_s,\ \mu_k,\ \mu_r$$

where F_n is the magnitude of the normal force and μ is the coefficient of friction. This coefficient has different values for static case (μ_s, gives the *maximum* force of friction when the body is still at rest), moving case (μ_k, for kinetic friction), and rolling case (μ_r).

IX. DRAG

When a body moves in a fluid, it encounters reaction forces depending on body's velocity $\boldsymbol{v}$ relative to the fluid. In many cases, one can model this phenomenon by assuming that the force is proportional only to a single power of the magnitude of velocity. The drag force $\boldsymbol{F}_D$ is always opposite to $\boldsymbol{v}$.

A. Linear drag

For small velocities, the dependence of force on velocity is linear

$$\boldsymbol{F}_D = -b\boldsymbol{v}$$

where b is an experimentally determined coefficient. We will encounter these forces when considering damped oscillations.

B. Quadratic drag

For larger velocities, the dependence is approximately quadratic

$$\boldsymbol{F}_D = -kAv^2\hat{\boldsymbol{v}}$$

where A is the cross-sectional area of the object perpendicular to the direction of motion and k is an experimentally determined coefficient. For motion in the air, $k = 0.25$ kg/m^3.

X. SYSTEMS OF BODIES

One often considers sets of bodies connected by massless ropes or rods and moving together. Such systems may contain ropes moving over massless pulleys whose only role is to change the direction of forces. The ropes exert the so-called tension forces denoted usually by $\boldsymbol{T}$. Since the ropes are massless, the tension forces on the ends of each rope have to be equal in magnitude. As bodies interact with each other, on free-body diagrams one should always identify pairs of "action-reaction" forces which are equal in magnitude.

As already stated in "Tricks of the trade" section, the best strategy for solving these types of problems is:

- Draw a "free-body" diagram. Make sure to identify the coordinate system (sometimes it is convenient to use a different coordinate system for each body).
- Write Newton's equations for each body: $m_i \boldsymbol{a}_i = \boldsymbol{F}_{\mathrm{net},i}$, $i = 1, \ldots, N$ making sure to include in $\boldsymbol{F}_{\mathrm{net},i}$ all the forces acting on body i.
- Identify action-reaction forces, in particular tension forces which are equal in magnitude.
- Identify components of $\boldsymbol{a}_i$ which are equal in magnitude since the bodies are moving together.
- Solve the resulting set of equations.

XI. CENTRIPETAL FORCES

The centripetal acceleration defined above has to be provided by a force. This so-called centripetal force acting on a body of mass m is given by

$$\boldsymbol{F}_c = m\boldsymbol{a}_c = \frac{mv^2}{r}(-\hat{\boldsymbol{r}}).$$

The centripetal forces can be tension, gravity, friction, or normal forces, as well as a combination of several such forces.

XII. LINEAR MOMENTUM AND IMPULSE

A. Definition of momentum

$$\boldsymbol{p} = m\boldsymbol{v}$$

B. Newton's equation in terms of momentum

Differentiating the definition of momentum,

$$\frac{d\boldsymbol{p}}{dt} = m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} \quad \text{or} \quad \frac{d\boldsymbol{p}}{dt} = \boldsymbol{F},$$

we get Newton's equation for linear momentum.

C. Definition of impulse

$$\boldsymbol{J}(t_0, t) = \int_{t_0}^{t} \boldsymbol{F}(t')\, dt'$$

D. Momentum-impulse theorem

$$\boldsymbol{p}(t) = \boldsymbol{p}(t_0) + \boldsymbol{J}(t_0, t).$$

To prove this theorem, integrate Newton's equation for momentum

$$\int_{t_1}^{t_2} \frac{d\boldsymbol{p}(t)}{dt} dt = \boldsymbol{p}(t_2) - \boldsymbol{p}(t_1) = \int_{t_1}^{t_2} \boldsymbol{F}(t)\, dt = \boldsymbol{J}(t_1, t_2).$$

E. Momentum conservation theorem

If the sum of all external forces acting on all bodies in the system $\boldsymbol{F}_{\text{ext}} = \sum_{i=1}^{N} \boldsymbol{F}_{\text{ext},i} = 0$, the total momentum of the system is conserved

$$\sum_{i=1}^{N} \boldsymbol{p}_i(t) = \text{const.}$$

i.e., it has a constant value in time. Notice that in these definitions we treat the vectors as if they were forming a single vector space, although they are really parts of different vector fields, e.g., $\boldsymbol{p}_i$ acts at the positions of particle i. To prove it for two particles in space without any external forces, we just add the Newton's equations for these particles

$$\frac{d\boldsymbol{p}_1}{dt} + \frac{d\boldsymbol{p}_2}{dt} = \boldsymbol{F}_{12} + \boldsymbol{F}_{21} = \boldsymbol{F}_{12} - \boldsymbol{F}_{12} = 0 \quad \Rightarrow \quad \boldsymbol{p}_1(t) + \boldsymbol{p}_2(t) = \text{const.},$$

where $\boldsymbol{F}_{ij}$ is the force exerted on particle i by particle j and we used the third Newton's law.

For N particles, we add Newton's equations for all particles. Then on the right-hand side we have

$$\boldsymbol{F} = \sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} \boldsymbol{F}_{ij} = \sum_{i=1}^{N} \sum_{j=1,\, j<i}^{N} \boldsymbol{F}_{ij} + \sum_{i=1}^{N} \sum_{j=1,\, j>i}^{N} \boldsymbol{F}_{ij},$$

where we just split each sum over j, which for a given i contains elements $j = 1, 2, \ldots, i-1, i+1, \ldots, N$ into the part up to $i-1$ and the part from $i+1$ to N. Since variables in summations are dummy, we can exchange i with j in the second sum:

$$\boldsymbol{F} = \sum_{i=1}^{N} \sum_{j=1,\, j<i}^{N} \boldsymbol{F}_{ij} + \sum_{j=1}^{N} \sum_{i=1,\, i>j}^{N} \boldsymbol{F}_{ji} = \sum_{i=1}^{N} \sum_{j=1,\, j<i}^{N} \boldsymbol{F}_{ij} - \sum_{j=1}^{N} \sum_{i=1,\, i>j}^{N} \boldsymbol{F}_{ij},$$

where we used the third Newton's law. Now we have to realize that both double sums, although they look different, go over the same set of pairs ij. First, see this for $N = 3$. For the first sum we have elements: 21, 31, 32, For the second sum, ji pairs are 12, 13, 23, so ij pairs are 21, 31, 32. Therefore, the sums cancel, proving the theorem. To generalize it to an arbitrary N, think of an $N \times N$ matrix. Both sums add all elements from the triangle below the diagonal: the first sum goes row by row, the second sum goes column by column.

XIII. WORK AND ENERGY

See Lecture Notes "Work and Energy" for a complete discussion of this subject.

A. Definition of work

The work done by a force $\boldsymbol{F}$ on a body moving along path C from point $\boldsymbol{r}_1$ to point $\boldsymbol{r}_2$ is given by the following line integral

$$W_{12,C} = \int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}. \tag{8}$$

The line integral is defined as follows (Riemann's definition)

$$\int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \lim_{N\to\infty} \sum_{i=0}^{N-1} \boldsymbol{F}(\boldsymbol{r}_i) \cdot \Delta\boldsymbol{r}_i \tag{9}$$

i.e., we divide the path C into N segments $\Delta\boldsymbol{r}_i$, project $\boldsymbol{F}$ computed at the beginning of the segment (actually, at an arbitrary point on the segment) on $\Delta\boldsymbol{r}_i$ (via the dot product), and compute the sum of all such contributions. Whereas the Riemann definition gives us a good intuitive understanding of line integrals, and forms the basis of various numerical methods of integral calculations, it does not tell us how to compute line integrals analytically. At 207/227 level, such integrals are evaluated only along straight-line segments along coordinate axes and in such cases they reduce to elementary integrals. A general method, described in calculus textbooks, parametrizes the path of integration. C

B. Definition of kinetic energy

$$K = \frac{mv^2}{2}$$

C. Work–kinetic energy theorem

$$\Delta K = K_2 - K_1 = W_{12,C},$$

where K_i it the kinetic energy at $\boldsymbol{r}_i$. This theorem is valid for any kinds of forces and independently of conservation of mechanical energy, subjects to be discussed below.

To prove this theorem, consider the following chain of equalities

$$\begin{aligned} W_{12,C} &= \int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = m \int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} \boldsymbol{a} \cdot d\boldsymbol{r} = m \int_{t_1}^{t_2} \left(\frac{d\boldsymbol{v}}{dt}\right) \cdot \left(\frac{d\boldsymbol{r}}{dt} dt\right) \\ &= m \int_{t_1}^{t_2} \left(\frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v}\right) dt = \frac{m}{2} \int_{t_1}^{t_2} \frac{d(\boldsymbol{v}\cdot\boldsymbol{v})}{dt} dt = \frac{m}{2} \int_{t_1}^{t_2} \frac{dv^2}{dt} dt \\ &= \frac{m}{2} v^2 \Big|_{t_1}^{t_2} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = K_2 - K_1 \end{aligned} \tag{10}$$

This theorem is derived from Newton's law, which was used in the second equality. We have also used in the derivation the identity

$$\frac{d}{dt}(\boldsymbol{v}\cdot\boldsymbol{v}) = \frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \frac{d\boldsymbol{v}}{dt} = 2(\frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v}).$$

D. Conservative forces

We define *conservative* force field as the field having the property that the integral (8) is independent of the path C.

Definition: A field is conservative if and only if

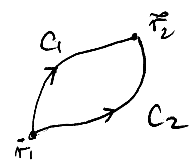
$$\int_{\boldsymbol{r}_1:C_1}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{\boldsymbol{r}_1:C_2}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} \tag{11}$$

for any two paths C_i.

This theorem is illustrated on the figure below. The reason for using the name "conservative" will be given later.

We will now prove the following auxiliary theorem

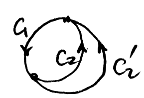
Theorem 1: The force $\boldsymbol{F}(\boldsymbol{r})$ is conservative $\Leftrightarrow \oint \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = 0$. The integral with the circle



symbol denotes a closed-loop line integral.
Proof (⇒):

$$\oint \boldsymbol{F}(\boldsymbol{r})\cdot d\boldsymbol{r} = \int_{\boldsymbol{r}_1:C_1}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r})\cdot d\boldsymbol{r} + \int_{\boldsymbol{r}_2:C_2}^{\boldsymbol{r}_1} \boldsymbol{F}(\boldsymbol{r})\cdot d\boldsymbol{r}$$
$$= \int_{\boldsymbol{r}_1:C_1}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r})\cdot d\boldsymbol{r} - \int_{\boldsymbol{r}_1:C_2}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r})\cdot d\boldsymbol{r} = 0 \tag{12}$$

since the field was assumed to be conservative.
Proof (⇐):



Consider the paths in the figure. We have two closed path integrals, one over the path $C_1 + C_2$ and another one over $C_1 + C_2'$. Each integral is zero by assumption. Since the integrals have a common part over C_1, the integral over C_2 must be equal to the integral over C_2'.

E. Existence of potential energy for conservative forces

If $\boldsymbol{F}(\boldsymbol{r})$ is conservative, there exists a function $U(\boldsymbol{r})$, called the potential energy. The difference of potential energies between two points is

$$U(\boldsymbol{r}_2) - U(\boldsymbol{r}_1) = -\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r})\cdot d\boldsymbol{r}. \tag{13}$$

Notice the minus sign. We can arbitrarily fix the value of U at some point $\boldsymbol{r}_0$ to get a unique function (we say that the potential energy is determined up to a constant).

The relation of Eq. (13) implies

$$\boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{\nabla} U(\boldsymbol{r}). \tag{14}$$

The existence of potential energy for a conservative force is a mathematical theorem. In most 207/227 textbooks this fact is introduced only via examples, see below. In some textbooks it is proved in one dimension. To carry out such a proof, consider the action of a force $\boldsymbol{F}$ on a path from x_1 to x_2. Since $d\boldsymbol{r} = dx\hat{\boldsymbol{x}}$, only $F_x(x)$ remains

$$W_{12} = \int_{x_1}^{x_2} F_x(x)dx = G(x_2) - G(x_1).$$

We did nothing else but used the simples definition of the integral, i.e., $G(x)$ is antiderivative of $F_x(x)$:

$$F_x(x) = \frac{dG(x)}{dx}.$$

Setting $U(x) = -G(x) + C$, we obtain one-dimensional equivalents of Eqs. (13) and (14)

$$U(x_1) - U(x_2) = \int_{x_1}^{x_2} F_x(x)dx \quad \text{and} \quad F_x(x) = -\frac{dU(x)}{dx}.$$

The only problem with this derivation is that it does not seem to depend on the force being conservative. However, it does. In calculus

$$\int_{x_1}^{x_2} F_x(x)dx = -\int_{x_2}^{x_1} F_x(x)dx.$$

Such F_x is conservative since the path $x_1 \to x_2 \to x_1$ is a closed path used to define the conservative force. If F_x is not conservative, for example it is a friction force making the two integrals in the equation above equal, our mathematical reasoning will not hold.

F. Examples of potential energy

Gravitational potential energy between two masses:

$$U_g = -G\frac{m_1 m_2}{r}$$

Gravitational potential energy near Earth's surface:

$$U_g = mgy$$

assuming the $\hat{\boldsymbol{y}}$ is vertical and points up.

Spring potential energy:

$$U = \frac{1}{2}kx^2,$$

where k is the spring constant (the constant in $F = -kx$).

XIV. MECHANICAL ENERGY CONSERVATION THEOREM

If all forces acting on a body are conservative, the sum of the kinetic and potential energies, called the total mechanical energy, is constant in time

$$T + U = E \equiv E_{\text{mech}} = \text{const.} \quad \text{or} \quad \Delta E_{\text{mech}} = \Delta K + \Delta U = 0.$$

Since $\Delta K = -\Delta U$, if U decreases by some amount, K increases by the same amount. This theorem is a direct conclusion from the work-kinetic energy theorem and the existence of the potential energy

$$K_2 - K_1 = \int_{\boldsymbol{r}_1:C_1}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = U_1 - U_2 \quad \Rightarrow \quad K_1 + U_1 = K_2 + U_2.$$

A. Mechanical plus thermal energy conservation law

If nonconservative (dissipative) forces are present, we have

$$\Delta E_{\text{mech}} = \Delta K + \Delta U = W_{12}^{\text{diss}},$$

where W_{12}^{diss} is the work done by the dissipative forces. Thus, the mechanical energy is not conserved in the presence of dissipative forces. So now, if U decreases by some amount, K increases by a smaller amount, and the difference is dissipated as heat. However, if we extend the energy concept to include the thermal energy, defined here as

$$\Delta E_{\text{th}} = -W_{12}^{\text{diss}},$$

it is possible to formulate the *energy conservation law*

$$\Delta E_{\text{tot}} = \Delta E_{\text{mech}} + \Delta E_{\text{th}} = 0 \quad \text{or} \quad E_{\text{tot}} = E_{\text{mech}} + E_{\text{th}} = \text{const.}$$

As stated earlier, if the total force is a sum of a conservative and nonconservative force, $\boldsymbol{F} = \boldsymbol{F}^c + \boldsymbol{F}^{nc}$, the work-kinetic energy theorem still holds. The work performed can be split into the conservative and nonconservative components. However, the kinetic energy cannot be split. This is because in an intermediate step of the proof in Eq. (10) we have the product $\boldsymbol{v} \cdot \boldsymbol{v}$ which produces mixed terms. Thus, the change of the kinetic energy is always due to the net force.

XV. COLLISIONS

A. Elastic collisions

Elastic collisions are collisions conserving the mechanical energy. In such cases and for linear collisions of two bodies, one can find the final velocities from initial velocities using momentum and energy conservation theorems

$$v_{1f,x} = \frac{m_1 - m_2}{m_1 + m_2}(v_{1i,x} - v_{2i,x}) + v_{2i,x} \qquad v_{2f,x} = \frac{2m_1}{m_1 + m_2}(v_{1i,x} - v_{2i,x}) + v_{2i,x}$$

B. Inelastic collisions

Inlastic collisions do not conserve mechanical energy. Therefore, we can use only the momentum conservation theorem. Thus, we need more information on a system to predict the outcome of such collisions.

XVI. ROTATION OF RIGID BODY

A. Kinematics of rotation

We consider only a special case of rotation of rigid bodies such that the rotation is around an axis fixed in space. Then each point of the body rotates in a plane perpendicular to the rotation axis (the planes are in general different for different points) on a circular orbit, with the same angular velocity for all points. Thus, all the angular kinematic equations for circular motion hold:

$$\omega(t) = \frac{d\theta}{dt}; \qquad \alpha(t) = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

However, linear quantities are different for different points since the radius of the circular motion, i.e., the distance of a point from the rotations axis, which we will denote by $\mathscr{r}$, is different for different points. For a point i

$$v_i = \omega \mathscr{r}_i \qquad a_{t,i} = \alpha \mathscr{r}_i.$$

B. Vector forms for ω and α

It is often convenient to represent ω and α by vector quantities. If one choses a coordinate system such that the rotation axis is along its $\hat{\boldsymbol{z}}$ axis, we can define

$$\boldsymbol{\omega} = \omega\hat{\boldsymbol{z}} \qquad \boldsymbol{\alpha} = \alpha\hat{\boldsymbol{z}}$$

with an additional assumption that ω and α are positive if rotations is counter-clock-wise (or according to right-hand rule) and negative otherwise.

Recalling Eqs. (5) and (6), we can write the relations

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{\mathscr{r}} \quad \text{and} \quad \boldsymbol{a}_t = \boldsymbol{\alpha} \times \boldsymbol{\mathscr{r}} \quad \text{or} \tag{15}$$

$$\boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{\mathscr{r}}_i \quad \text{and} \quad \boldsymbol{a}_{t,i} = \boldsymbol{\alpha} \times \boldsymbol{\mathscr{r}}_i, \tag{16}$$

which result from orthogonality of the vectors involved.

C. Center of mass

Consider first a model of rigid body consisting of point masses m_i, $i = 1, \ldots, N$ connected by rigid massless rods. The center of mass (CM) of such system is defined as

$$\boldsymbol{R}_{\text{CM}} = \frac{1}{M} \sum_{i=1}^{N} \boldsymbol{r}_i m_i,$$

where M is the sum of all masses. To understand this concept, first consider two bodies of equal mass. Clearly, $\boldsymbol{R}_{\text{CM}}$ is in the midpoint between these bodies. Now consider the case when one of the masses is much larger than the other one. $\boldsymbol{R}_{\text{CM}}$ is now very close to the large mass and still on the line connecting the two objects. For example, the CM of the Earth-Moon system is 4,600 km from the center of the Earth.

For a continuous rigid body, this definition becomes

$$\boldsymbol{R}_{\text{CM}} = \frac{1}{M} \int_{\text{body}} \boldsymbol{r}\, dm = \frac{1}{M} \int_{\text{body's volume}} \boldsymbol{r} \rho \, dV$$

where ρ is the density at point $\boldsymbol{r}$. Note that $\boldsymbol{r}$ can be measured in any coordinate system and the CM vector is then expressed in the same coordinate system. Also, $\boldsymbol{r}$ here is the regular positions vector, not to be confused with the $\mathscr{r}$ defined above.

D. Moment of inertia

Analogously, the moment of inertia is defined as

$$I = \sum_{i=1}^{N} \mathscr{r}_i^2 m_i,$$

or

$$I = \int_{\text{body}} \mathscr{r}^2 dm = \int_{\text{body's volume}} \mathscr{r}^2 \rho dV$$

E. Parallel axes theorem

The moments of intertia defined with respect to different parallel axes are related:

$$I = I_{\text{CM}} + Md^2,$$

where I_{CM} is the moment of inertia relative to an axis through the center of mass and I is the moment of inertia relative to an axis parallel to it and displaced by a distance d. This theorem is proved in the following way. Consider a point within the body such that its distance vector from the axis through CM is $\vec{\mathscr{r}}$ and from the new axis is $\vec{\mathscr{r}}'$. Both vectors are in the same plane perpendicular to both axes. We have then

$$\vec{\mathscr{r}} = \boldsymbol{h} + \vec{\mathscr{r}}',$$

where $\boldsymbol{h}$ is the vector from the axis through CM to the new one, so that

$$\mathscr{r}'^2 = \mathscr{r}^2 + h^2 - 2\boldsymbol{h} \cdot \vec{\mathscr{r}}.$$

Integrating over all points, we get

$$I = \int_{\text{body}} \mathscr{r}'^2 dm = I_{\text{CM}} + h^2 \int_{\text{body}} dm - 2\boldsymbol{h} \cdot \int_{\text{body}} \vec{\mathscr{r}} dm$$

Assume that the CM is located at the center of coordinate systems, i.e., $\boldsymbol{R}_{\text{CM}} = (0, 0, 0)$. The last integral is a part of the integral defining $\boldsymbol{R}_{\text{CM}}$:

$$\boldsymbol{R}_{\text{CM}} = \frac{1}{M} \int_{\text{body}} \left(\vec{\mathscr{r}} + z\hat{\boldsymbol{z}}\right) dm = (0, 0, 0).$$

This proves the theorem.

F. Definition of torque

The torque is defined as

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = rF \sin\alpha \, \hat{\boldsymbol{a}}_{\perp},$$

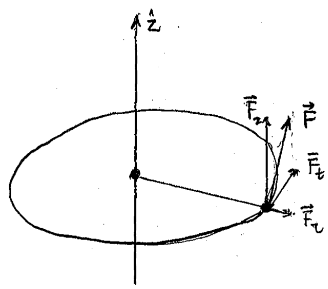
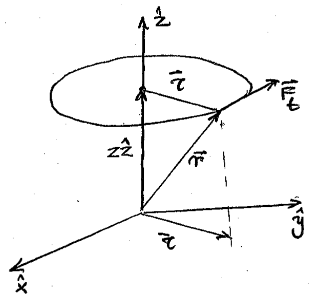
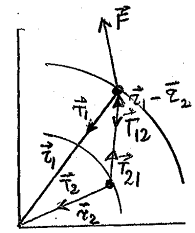
where α is the angle between the vectors and $\hat{\boldsymbol{a}}_{\perp}$ is a unit vector perpendicular to the plane spanned by $\boldsymbol{r}$ and $\boldsymbol{F}$ and oriented according to the right-hand rule.

G. Rotational Newton's equation

A rigid body rotating around an axis fixed in space obeys the following rotational Newton's equation (called also rotational Newton's theorem or law):

$$I\boldsymbol{\alpha} = \boldsymbol{\tau},$$

where $\boldsymbol{\tau}$ is the torque along the axis of rotation. To derive this formula from the standard Newton's equation, first consider a single point-like mass m rotating around an axis fixed in space and connected to this axis by a weightless stiff rod, see the figure. The external force $\boldsymbol{F}$ acting on this body can be pointing in arbitrary direction. First, decompose this force into the tangential, $\hat{\boldsymbol{\theta}}$, radial, $\hat{\boldsymbol{\mathscr{r}}}$, and vertical, $\hat{\boldsymbol{z}}$, components. All components but the

tangential one are balanced by the reaction forces of the system: tension and stiffness of rod attached to the rotation axis fixed in space. Thus, Newton's equation for the particle is

$$m\,\boldsymbol{a}_t = \boldsymbol{F}_t \quad \text{or} \quad m\,\boldsymbol{\alpha} \times \vec{\mathscr{r}} = \boldsymbol{F}_t, \tag{17}$$

where we used the vector form of the angular acceleration introduced in Sec. XVI B.

Now consider the same system but, to determine torques, describe the position of the particle by the vector $\boldsymbol{r}$ from the center of coordinate system (see the figure)

$$\boldsymbol{r} = \vec{\mathscr{r}} + z\hat{\boldsymbol{z}}.$$

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}_t = (\vec{\mathscr{r}} + z\hat{\boldsymbol{z}}) \times \boldsymbol{F}_t = \vec{\mathscr{r}} \times \boldsymbol{F}_t + z\hat{\boldsymbol{z}} \times \boldsymbol{F}_t = \tau_z\hat{\boldsymbol{z}} + \tau_{\mathscr{r}}\hat{\mathscr{r}}.$$

The first cross product is oriented along the z axis. The second one is oriented along the $\hat{\mathscr{r}}$ and it is not zero. However, this torque action is to rotate the axis of rotation of the system, which is fixed in space. Thus, there is reaction torque from the system that is opposite to $\tau_{\mathscr{r}}\hat{\mathscr{r}}$ and cancels it

$$\boldsymbol{\tau}_{\rm net} = \tau_z\hat{\boldsymbol{z}} + \tau_{\mathscr{r}}\hat{\mathscr{r}} + \boldsymbol{\tau}_{\rm reaction} = \tau_z\hat{\boldsymbol{z}}.$$

We will omit the two torques that cancel in further developments.

To express Eq. (17) in terms of torque, multiply it by $\vec{\mathscr{r}}$ from the left (we do not multiply by $\boldsymbol{r} = \vec{\mathscr{r}} + z\hat{\boldsymbol{z}}$ since we have already eliminated the action of the second term from considerations):

$$m\vec{\mathscr{r}} \times (\boldsymbol{\alpha} \times \vec{\mathscr{r}}) = \vec{\mathscr{r}} \times \boldsymbol{F}_t = \tau_z\hat{\boldsymbol{z}}.$$

The double vector product can be simplified using the BAC-CAB rule

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}). \tag{18}$$

So that we get

$$m\vec{\mathscr{r}} \times (\boldsymbol{\alpha} \times \vec{\mathscr{r}}) = m[\boldsymbol{\alpha}(\vec{\mathscr{r}} \cdot \vec{\mathscr{r}}) - \vec{\mathscr{r}}(\boldsymbol{\alpha} \cdot \vec{\mathscr{r}})] = m\mathscr{r}^2\boldsymbol{\alpha}$$

The second dot product is zero since the two vectors are perpendicular. In this way we get rotational Newton's equation for a single particle

$$m\mathscr{r}^2\boldsymbol{\alpha} = \tau_z\hat{\boldsymbol{z}}. \tag{19}$$

We first generalize our considerations to a set of 2 particles with masses m_i at positions $\boldsymbol{r}_i$. The particles are connected by a set of rigid weightless rods. First assume that there are external forces, in general different, acting on each particle. We cannot immediatelly write Eq. (19) for each particle since the reaction forces from other particles and rods may affect the torques. Instead of considering the case of forces acting on each particle, let us assume that the external force acts only on particle 1. This is, in fact, a more relevant case to consider for rotation of a rigid body since the external force is in most cases acting on only one place at the surface. Also, assume that the two bodies lie in a plane perpendicular to the axis of rotation, see the figure. The tension forces from the rods connecting particles to the axis of rotation do not contribute to the torque since they act along $\vec{\boldsymbol{r}}_i$. Consider now the tensions acting along the rod connecting the two particles. Since the rod is weightless, the two forces are of equal magnitude and opposite direction. Each of these forces exerts a torque, but the sum of these torques

$$\vec{\boldsymbol{r}}_1 \times \boldsymbol{T}_{12} + \vec{\boldsymbol{r}}_2 \times \boldsymbol{T}_{21} = (\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2) \times \boldsymbol{T}_{12} = 0$$

since the difference vector is along $\boldsymbol{T}_{12}$, see the figure. Thus, the only force that contributes to the torque is $\boldsymbol{F}$ or, from the considerations above, its tangential component. We can now write the two equations

$$\begin{aligned} m_1 \vec{\boldsymbol{r}}_1^2 \boldsymbol{\alpha} &= \vec{\boldsymbol{r}}_1 \times \boldsymbol{F}_t + \vec{\boldsymbol{r}}_1 \times \boldsymbol{T}_{12} \\ m_2 \vec{\boldsymbol{r}}_2^2 \boldsymbol{\alpha} &= \vec{\boldsymbol{r}}_2 \times \boldsymbol{T}_{21} \end{aligned} \tag{20}$$

and add them, getting

$$(m_1 \vec{\boldsymbol{r}}_1^2 + m_2 \vec{\boldsymbol{r}}_2^2)\boldsymbol{\alpha} = \vec{\boldsymbol{r}}_1 \times \boldsymbol{F}_t \tag{21}$$

since the torques from tension forces cancel. Clearly, this consideration can be extended to any number of particles since we will have cancellations of torques in each pair of particles. We can write therefore

$$\sum_i^N m_i \vec{\boldsymbol{r}}_i^2 \boldsymbol{\alpha} = \sum_i^N \tau_{z,i} \hat{\boldsymbol{z}} = \vec{\boldsymbol{r}}_1 \times \boldsymbol{F}_t, \tag{22}$$

Assuming still that the external force acts only on body 1.

To extend to a rigid body, we divide the body into small masses Δm_i. As discussed earlier, all these masses accelerate with the same angular acceleration. For each mass, we can rewrite Eq. (19) as

$$\Delta m_i \vec{\boldsymbol{r}}_i^2 \boldsymbol{\alpha} = \tau_{z,i} \hat{\boldsymbol{z}}, \tag{23}$$

so that adding all the contributions, we get

$$\sum_{i}^{N} \Delta m_i \, r_i^2 \boldsymbol{\alpha} = \sum_{i}^{N} \tau_{z,i} \hat{\boldsymbol{z}} == \vec{\boldsymbol{r}} \times \boldsymbol{F}_t = \tau_z \hat{\boldsymbol{z}}, \tag{24}$$

where $\vec{\boldsymbol{r}}$ is the point where $\boldsymbol{F}$ acts. Going to the limit of infinitesimally small masses, one gets the appropriate integral

$$\left[\int_{\text{body}} r_i^2 dm \right] \boldsymbol{\alpha} = I \boldsymbol{\alpha} = \tau_z \hat{\boldsymbol{z}}. \tag{25}$$

We can now write

$$I \boldsymbol{\alpha} = \tau_z \hat{\boldsymbol{z}} \quad \text{or} \quad I \alpha = \tau, \tag{26}$$

where in the last equation we tacitly assume that the quantities refer to the axis of rotation.

H. Definition of angular momentum for a set of particles

For a point mass particle, the angular momentum, $\boldsymbol{l}$, is defined as:

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p}.$$

This definition holds for any type of motion, including motion along straight line. For a set of particles (not necessarily connected in any way) the total $\boldsymbol{L}$ is defined as

$$\boldsymbol{L} = \sum_{i=1}^{N} \boldsymbol{l}_i.$$

Again, the particles are not necessarily rotating.

I. Newton's equation for angular momentum for a set of particles

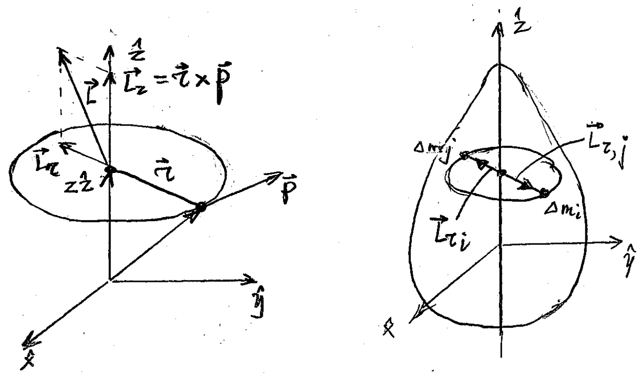
For a single particle, starting from the time derivative of $\boldsymbol{l}$ and using Newton's equation, we get

$$\frac{d\boldsymbol{l}}{dt} = m \left[\frac{d\boldsymbol{r}}{dt} \times \boldsymbol{v} + \boldsymbol{r} \times \frac{d\boldsymbol{v}}{dt} \right] = m\boldsymbol{r} \times \boldsymbol{a} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{\tau},$$

since $\boldsymbol{v} \times \boldsymbol{v} - 0$. The final equation,

$$\frac{d\boldsymbol{l}}{dt} = \boldsymbol{\tau},$$

is called Newton's equation (or law) for angular momentum. We have used in this derivation the fact that $\boldsymbol{v} \times \boldsymbol{v} = 0$. In a general case, the vector $\boldsymbol{l}$ is not pointed along any direction, but changes the direction in time. Even if the particle is in a circular motions around an



axis fixed in space along $\hat{\boldsymbol{z}}$ axis, the angular momentum is not directed along this axis, see the figure.

This expression is immediately extended to N particles

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{\tau}, \tag{27}$$

where $\boldsymbol{\tau} = \sum_{i=1}^{N} \boldsymbol{\tau}_i$. Note that here, in contrast to the derivation of Newton's equations for rotation of rigid body, each particle is equivalent and is being acted on by some external force. Only the external forces contribute to torques, for the same reasons as discussed for the rigid body.

J. Angular momentum conservation

For a single particle if $\boldsymbol{\tau} = 0$ then

$$\frac{d\boldsymbol{l}}{dt} = 0 \quad \Rightarrow \quad \boldsymbol{l} = \text{const.}$$

The torque can be zero if the net force acting on a body is zero, which is not an interesting case. The other possibility is that the net force $\boldsymbol{F}$ is parallel to $\boldsymbol{r}$.

Similarly, if the total torque $\boldsymbol{\tau} = 0$ then

$$\boldsymbol{L}(t) = \text{const.}$$

Here in addition to the cases for a torque be zero for a single particle, the torque is zero if the sum of all torques due to the external forces is zero. The reason is that the torques from internal forces cancel out, similarly as in the case of the linear momentum conservation the internal forces cancel out, as proved in Sec. XII E. This fact will be shown in 419. An important case is when there are no external forces acting on a system of particles, i.e., the system is isolated. For isolated systems both the linear and angular momentum are always conserved.

K. Example: proof of second Kepler's law

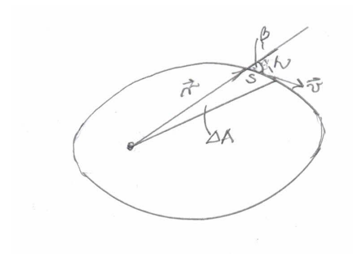
In Sec. VI D we have formulated the second Kepler's law:

The vector from the sun to the planet sweeps equal areas in equal times

Proof: Assume that the Sun is infinitely heavy and place it at the center of coordinate system. Consider a planet that moves in Sun's gravitational field. Since the position vector of the planet $\boldsymbol{r}$ is along the force of gravity $\boldsymbol{F}$, the torque $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = 0$. Thus, the angular momentum $\boldsymbol{l}$

$$|\boldsymbol{l}| = |\boldsymbol{r} \times \boldsymbol{p}| = m|\boldsymbol{r} \times \boldsymbol{v}| = mrv\sin\beta,$$

where m is the mass of the planet, $\boldsymbol{v}$ is its linear velocity, and β is the angle between $\boldsymbol{r}$ and $\boldsymbol{v}$ (see the figure), is conserved. The area swept in a short time Δt is the area of the triangle ΔA. Since the height of this triangle is $h = \Delta s\, \sin\beta$, where Δs is the distance traveled by the planet in Δt,



$$\Delta A = \frac{1}{2} r\,\Delta s \sin\beta \quad \text{and} \quad \frac{\Delta A}{\Delta t} = \frac{1}{2} rv\sin\beta = \frac{l}{2m}.$$

Since the right-hand side is constant, this proves the theorem. Notice that this proof does not make any assumption about the orbit, i.e., holds for any shape orbits and any central forces.

L. Definition of angular momentum for a symmetric rigid body

As before, we assume that the rigid body rotates about an axis fixed in space. By analogy with a set of particles connected by rigid rods we can define the angular momentum of this body as

$$\boldsymbol{L} = \int_{\text{body}} \boldsymbol{r} \times \boldsymbol{v}\, dm.$$

In contrast to the analysis of the rotation of a rigid body around an axis fixed in space (assumed to be along $\hat{\boldsymbol{z}}$ axis) in terms of its angular velocity, we cannot neglect the components of $\boldsymbol{L}$ in directions other than $\hat{\boldsymbol{z}}$ (as it was possible for torques due to reaction forces and the third Newton's law). Therefore, we will assume that the body is symmetric with respect to the $\hat{\boldsymbol{z}}$. This means that for any mass Δm_i in a plane perpendicular to $\hat{\boldsymbol{z}}$ there is another mass such that $\Delta m_i = \Delta m_j$ and the two masses are on the line through the axis at equal distances from the axis. In such a case, the angular momenta components of these two masses along $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ cancel, see the figure in Sec. XVI I

$$\boldsymbol{L}_i = (\vec{\boldsymbol{r}}_i + z_i\hat{\boldsymbol{z}}) \times \boldsymbol{p}_i = L_{z,i}\hat{\boldsymbol{z}} + L_{r,i}\,\hat{\boldsymbol{r}}_i \tag{28}$$

Thus, for a symmetric body rotating around $\hat{\boldsymbol{z}}$

$$\boldsymbol{L} = \int_{\text{body}} \boldsymbol{r} \times \boldsymbol{v}\, dm = L_z \hat{\boldsymbol{z}}.$$

M. Kinematics of angular momentum of symmetric rigid body

We can rewrite Eq. (28), keeping only the $\hat{\boldsymbol{z}}$ component, using the vector formula for $\boldsymbol{v}_i$

$$L_{z,i}\hat{\boldsymbol{z}} = m\vec{\boldsymbol{\imath}}_i \times \boldsymbol{v} = m\vec{\boldsymbol{\imath}}_i \times (\boldsymbol{\omega} \times \vec{\boldsymbol{\imath}}_i) = m\boldsymbol{\omega}(\vec{\boldsymbol{\imath}}_i \cdot \vec{\boldsymbol{\imath}}_i) + m\vec{\boldsymbol{\imath}}_i(\boldsymbol{\omega} \cdot \vec{\boldsymbol{\imath}}_i) = m\boldsymbol{\imath}_i^2\boldsymbol{\omega}$$

where we used Eq. (18) and the fact that $\vec{\boldsymbol{\imath}}_i$ and $\boldsymbol{\omega}$ are orthogonal. We can now sum over i and then change the sum into integration, obtaining in this way the kinematic relation between the total angular momentum and angular velocity:

$$L_z\hat{\boldsymbol{z}} = I\boldsymbol{\omega}. \tag{29}$$

We say that this relation is kinematic since we did not use Newton's law in its derivation.

N. Newton's equation for angular momentum of symmetric rigid body

Equation (27) includes summation over a set of arbitrary particles. We will first apply this equations to a set of particles connected by rigid weightless rods. Next, we assume this system rotates around an axis fixed in space and is symmetric with respect to this axis. Then, we will assume that the external force acts only on one particle on the outside of the body. Finally, we will change summation into integration. With all these assumptions, the left-hand side becomes the total L_z component as in Eq. (29), while while the right-hand side becomes the total torque τ_z:

$$\frac{dL_z}{dt}\hat{\boldsymbol{z}} = \tau_z\hat{\boldsymbol{z}}.$$

This is Newton's equation for angular momentum of symmetric rigid body.

O. Comparison of rotational expressions

We now have three expressions relating torques, angular velocities, angular accelerations, and moments of inertia

$$\boldsymbol{L} = I\boldsymbol{\omega}; \qquad I\boldsymbol{\alpha} = \boldsymbol{\tau}; \qquad \frac{d\boldsymbol{L}}{dt} = \boldsymbol{\tau}$$

They are, of course consistent. For example, equate the second and the third equations

$$\frac{d\boldsymbol{L}}{dt} = I\boldsymbol{\alpha}$$

and differentiate the first equations

$$\frac{d}{dt}\boldsymbol{L} = \frac{d}{dt}\left[I\boldsymbol{\omega}\right] \quad \Rightarrow \quad \frac{d\boldsymbol{L}}{dt} = I\boldsymbol{\alpha}$$

to get the same expression.

P. Rotational kinetic energy

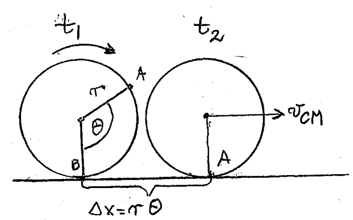
For a rigid body (does not need to be symmetric), the kinetic energy of a small mass Δm_i within the body, such that $M = \sum_{i=1}^{N} \Delta m_i$, is

$$K_i = \frac{1}{2}\Delta m_i v_i^2 = \frac{1}{2}\Delta m_i (\boldsymbol{\omega} \times \vec{r}_i)^2 = \frac{1}{2}\Delta m_i r_i^2 \omega^2,$$

where in the last equality we used the fact that $\vec{r}_i$ is perpendicular to $\boldsymbol{\omega}$. Summing over i and then going with N to infinity, we get the integral defining the moment of inertia I, so that

$$K = \frac{I\omega}{2}.$$

Q. Rolling without slipping



Rolling without slipping is an important phenomenon in everyday life. Consider a wheel of radius r in such motion (see the picture). In the frame of the wheel, the rotation proceed with a constant angular velocity $\omega = d\theta/dt$ and the magnitude of the velocity of a point on the rim is $v = r\omega$. Since there is no slipping, at each instant of time point B is at rest with respect to the ground (otherwise, there would be slipping). Thus, as the wheel rotates by angle θ, the distance covered by the center of mass of the wheel is the same as the length of the arch spanned by θ: $\Delta x = r\theta$. Thus,

$$v_{\text{CM}} = v = r\omega.$$

XVII. OSCILLATIONS

A. Simple harmonic oscillator

Simple harmonic oscillator is a one-dimensional system with the force $F = -kx$, where k is called the spring constant. Newton's equation for such system

$$m\frac{d^2x(t)}{dt^2} = -kx(t), \tag{30}$$

has solution

$$x(t) = A\cos(\omega_0 t + \phi) \tag{31}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}$$

and A (amplitude) and ϕ (phase shift) are arbitrary constants which can be fixed using initial conditions

$$x(t_0) = x_0 \quad \text{and} \quad v(t_0) = v_0.$$

Instead of deriving this solutions, we just check that it satisfies Eq. (30).

B. Damped harmonic oscillator

Newton's equation

$$m\frac{d^2x(t)}{dt^2} = -kx(t) - b\frac{d\,x(t)}{dt},$$

where b is the damping constant, has solution

$$x(t) = A\mathrm{e}^{-t/2\tau}\cos(\omega t + \phi) \quad \text{where} \quad \omega = \sqrt{\frac{k}{m} - \frac{1}{4\tau^2}},$$

the time constant $\tau = m/b$, and A and ϕ are adjustable parameters. The time-dependent amplitude can be defined as the maximum value of x in a given oscillation, corresponding to a turning point where the velocity is zero. The envelope of the motion is defined as

$$x_{\mathrm{env}}(t) = A\mathrm{e}^{-t/2\tau}.$$

Note that the envelope is equal to $x(t)$ only at the points where $\cos(\omega t + \phi) = 1$ which are different for the positions of the maxima of $x(t)$.

C. Forced (driven) damped harmonic oscillator

If there is an additional force acting on the particle $F_{\mathrm{drive}} = F_d\cos(\omega_2 t)$, the Newton equation is

$$m\frac{d^2x(t)}{dt^2} = -kx(t) - b\frac{d\,x(t)}{dt} + F_d\cos(\omega_2 t). \tag{32}$$

The solution for arbitrary time is complicated, but one can show that after a sufficient time the initial (transient) solution dies off and the steady state solution is

$$x(t) = \frac{F_d}{G}\cos(\omega_2 t - \beta), \tag{33}$$

where

$$G = \sqrt{m^2(\omega_2^2 - \omega_0^2)^2 + b^2\omega_2^2} \quad \text{and} \quad \beta = \cos^{-1}\frac{b\omega_2}{G}.$$

The remaining sections are not reviewed in class since they are not covered in the present PHYS 419

XVIII. STATIC FLUIDS

Pressure is defined as the ration of the total force acting on a surface to the area of the surface

$$p = \frac{F}{A}.$$

A. Hydrostatic theorem

Hydrostatic theorem states that for a uniform fluid

$$p = p_0 + \rho g d = \text{const.},$$

where p_0 is atmospheric pressure and d is the depth in the fluid.

B. Buoyancy force (Archimedes law)

$$F_b = g\rho_f V_f$$

where ρ_f is the density and V_f is the volume of the displaced fluid.

XIX. FLUID DYNAMICS

We consider a nonviscous, noncompressible fluid in laminar flow. Consider two points along a "flow tube". The velocity, cross-sectional area of the tube, pressure, and height of each point are v_i, A_i, p_i, and y_i, respectively.

A. Continuity equation

$$v_1 A_1 = v_2 A_2$$

B. Bernoulli's equation

$$p_1 + \frac{1}{2}\rho v_1^2 + g\rho y_1 = p_2 + \frac{1}{2}\rho v_2^2 + g\rho y_2.$$

XX. ELASTICITY

A. Linear distortion

$$\frac{F}{A} = Y\frac{\Delta L}{L}$$

where F is force, A is the cross-sectional area of the rod, L is the length of the rod, and ΔL is the elongation. The Young modulus $Y = kL/A$ where k is the "spring constant" for the rod.

B. Volume distortion

$$\frac{F}{A} = -B\frac{\Delta V}{V}$$

where A is the surface area of the volume V and B is called bulk modulus.