PHYS 419: Classical Mechanics Lecture Notes
POLAR COORDINATES

A vector in two dimensions can be written in Cartesian coordinates as

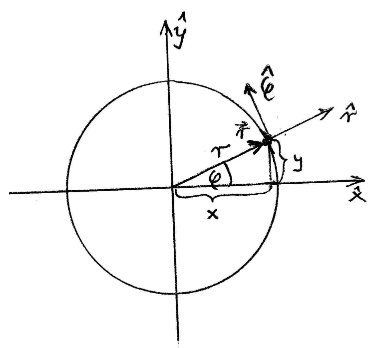
$$\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} \tag{1}$$

where $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are unit vectors in the direction of Cartesian axes and x and y are the components of the vector, see also the figure. It is often convenient to use coordinate systems other than the Cartesian system, in particular we will often use polar coordinates. These coordinates are specified by $r = |\boldsymbol{r}|$ and the angle ϕ between $\boldsymbol{r}$ and $\hat{\boldsymbol{x}}$, see the figure. The relations between the polar and Cartesian coordinates are very simple:

$$x = r\cos\phi \qquad y = r\sin\phi$$

and

$$r = \sqrt{x^2+y^2} \qquad \phi = \arctan\frac{y}{x}.$$



The unit vectors of polar coordinate system are denoted by $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$. The former one is defined accordingly as

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r} \tag{2}$$

Since

$$\boldsymbol{r} = r\cos\phi\ \hat{\boldsymbol{x}} + r\sin\phi\ \hat{\boldsymbol{y}},$$

$$\hat{\boldsymbol{r}} = \cos\phi\ \hat{\boldsymbol{x}} + \sin\phi\ \hat{\boldsymbol{y}}.$$

The simplest way to define $\hat{\boldsymbol{\phi}}$ is to require it to be orthogonal to $\hat{\boldsymbol{r}}$, i.e., to have $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\phi}} = 0$. This gives the condition

$$\cos\phi\ \phi_x + \sin\phi\ \phi_y = 0.$$

The simplest solution is $\phi_x = -\sin\phi$ and $\phi_y = \cos\phi$ or a solution with signs reversed. This gives

$$\hat{\boldsymbol{\phi}} = -\sin\phi\ \hat{\boldsymbol{x}} + \cos\phi\ \hat{\boldsymbol{y}}.$$

This vector has unit length

$$\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \sin^2\phi + \cos^2\phi = 1.$$

The unit vectors are marked on the figure. With our choice of sign, $\hat{\boldsymbol{\phi}}$ points in the direction of increasing angle ϕ. Notice that $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ are drawn from the position of the point considered. Notice also that due to Eq. (2), the expression for $\boldsymbol{r}$ in terms of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ is

$$\boldsymbol{r} = r\hat{\boldsymbol{r}}.$$

An expression analogous to Eq. (1) is wrong

$$\boldsymbol{r} \neq r\hat{\boldsymbol{r}} + \phi\hat{\boldsymbol{\phi}}.$$

We will need also the derivatives of vector $\boldsymbol{r}$ expressed in polar coordinates. We have

$$\dot{\boldsymbol{r}} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\hat{\boldsymbol{r}}}$$

and

$$\dot{\hat{\boldsymbol{r}}} = \frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} = -\dot{\phi}\sin\phi\ \hat{\boldsymbol{x}} + \dot{\phi}\cos\phi\ \hat{\boldsymbol{y}} = \dot{\phi}\left(-\sin\phi\ \hat{\boldsymbol{x}} + \cos\phi\ \hat{\boldsymbol{y}}\right) = \dot{\phi}\hat{\boldsymbol{\phi}}$$

(notice that in contrast to Cartesian coordinate system, derivatives of unit vectors of the polar system are not zero) so that

$$\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}.$$

Now get the second derivative

$$\ddot{\boldsymbol{r}} = \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\dot{\hat{\boldsymbol{r}}} + \dot{r}\dot{\phi}\hat{\boldsymbol{\phi}} + r\ddot{\phi}\hat{\boldsymbol{\phi}} + r\dot{\phi}\dot{\hat{\boldsymbol{\phi}}},$$

so that the only new derivative is that of $\hat{\boldsymbol{\phi}}$:

$$\dot{\hat{\boldsymbol{\phi}}} = \dot{\phi}\left(-\cos\phi\ \hat{\boldsymbol{x}} - \sin\phi\ \hat{\boldsymbol{y}}\right) = -\dot{\phi}\hat{\boldsymbol{r}}.$$

Grouping terms together, we finally get:

$$\ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{\boldsymbol{r}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\boldsymbol{\phi}}.$$