PHYS 419: Classical Mechanics Lecture Notes
MOTION IN CONSTANT MAGNETIC FIELD

As another example of applications of Newton's laws, consider a particle of charge q moving in a constant magnetic field oriented along the $\hat{\boldsymbol{z}}$ axis

$$\boldsymbol{B} = B\hat{\boldsymbol{z}}.$$

There are no other fields (in particular, there is no gravity). The initial conditions are

$$\boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{and} \quad \boldsymbol{r}(0) = \boldsymbol{r}_0.$$

The Lorentz force acting on the particle is

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} = q \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = q(Bv_y, -Bv_x, 0)$$

Newton's equation is:

$$m\ddot{\boldsymbol{r}} = q\boldsymbol{v} \times \boldsymbol{B}$$

or in component form

$$m\ddot{x} = qBv_y$$

$$m\ddot{y} = -qBv_x$$

$$m\ddot{z} = 0$$

Introducing the notation

$$\omega = \frac{qB}{m}$$

we can write the equations in a compact form

$$\dot{v}_x = \omega v_y$$

$$\dot{v}_y = -\omega v_x$$

One of the most efficient ways to solve these equations is to use complex functions. Let us define

$$\eta(t) = v_x(t) + iv_y(t),$$

calculate the derivative of this function, and use Newton's equations:

$$\dot{\eta} = \dot{v_x} + \dot{v_y} = \omega v_y - i\omega v_x = -i\omega(v_x + iv_y) = -i\omega\eta.$$

The resulting form of Newton's equation

$$\dot{\eta} = -i\omega\eta$$

suggests that the solution is an exponential function. Let us check it

$$\frac{d\mathrm{e}^{i\omega t}}{dt} = \frac{d}{dt}(\cos\omega t + i\sin\omega t) = \omega(-\sin\omega t + i\cos\omega t) = i\omega(\cos\omega t + i\sin\omega t) = i\omega\mathrm{e}^{i\omega t}$$

Therefore,

$$\eta(t) = A\mathrm{e}^{-i\omega t}$$

where A is an arbitrary constant, solves our equation. The constant can be determined from the initial conditions:

$$\eta(0) = A = v_{x0} + iv_{y0}$$

Our problem is now essentially solved, but when we use complex functions, in the last step we have to extract from the complex solution the real solution, since measurable quantities are always real. To do so, represent the complex constant A in the trigonometric form

$$A = a\mathrm{e}^{i\delta}$$

where $a = \sqrt{v_{x0}^2 + v_{y0}^2}$ and $\delta = \arctan(v_{y0}/v_{x0})$ are real. Now the solution can be written as

$$\eta(t) = a\mathrm{e}^{i\delta}\mathrm{e}^{-i\omega t} = a\mathrm{e}^{-i(\omega t - \delta)}$$

or

$$\eta(t) = a\cos(\omega t - \delta) - ia\sin(\omega t - \delta)$$

Taking into account the initial definition of η, we get the real solutions:

$$v_x(t) = a\cos(\omega t - \delta) \qquad v_y(t) = -a\sin(\omega t - \delta)$$

To get $\boldsymbol{r}(t)$, integrate $\eta(t) = \dot{\zeta}(t) = \dot{x}(t) + i\dot{y}(t)$

$$\zeta(t) = \int d\eta = A\int \mathrm{e}^{-i\omega t}dt + C = -\frac{A}{i\omega}\mathrm{e}^{-i\omega t} + C = \frac{iA}{\omega}\mathrm{e}^{-i\omega t} + C$$

Using real constants and the initial conditions we get

$$\zeta(t) = \frac{ia}{\omega}\mathrm{e}^{-i(\omega t - \delta)} + C$$

$$\zeta(0) = x_0 + iy_0 = \frac{ia}{\omega}\mathrm{e}^{-i\delta} + C$$

and therefore the constant is

$$C = x_0 + iy_0 - \frac{ia}{\omega}\mathrm{e}^{-i\delta}.$$

The real form of the position vector is therefore

$$x(t) = \frac{a}{\omega}\sin(\omega t - \delta) + x_0 + \frac{a}{\omega}\sin\delta$$

$$y(t) = \frac{a}{\omega}\cos(\omega t - \delta) + y_0 - \frac{a}{\omega}\cos\delta$$

The resulting motion can be easily intepreted if we look at it in the coordinate system centered at

$$(x_0 + \frac{a}{\omega}\sin\delta, y_0 - \frac{a}{\omega}\cos\delta).$$

If $v_{z0} = 0$, the motion is circular around this point. If $v_{z0} \neq 0$, the motion forms a helix.

One of the most spectacular manifestations of this motion are polar lights (aurora). Cosmic charged particles move around helical lines in the Earth magnetic field. Their collisions with air particles produce the lights.