PHYS 419: Classical Mechanics Lecture Notes
COMPLEX NUMBERS

Complex numbers and functions of these numbers are often used in physics. In fact, functional analysis attains its full power only over the complex plane. We will not use any advanced concepts from functional analysis in this course, but elementary applications of complex numbers will be very handy.

A complex number z is defined just as an ordered pair of real numbers

$$z = (x, y)$$

where x is called the real component and y the imaginary component. We sometimes use the notation

$$x = \Re(z) = \mathcal{R}e(z) \quad y = \Im(z) = \mathcal{I}m(z)$$

Thus, it can be represented in a plane in the same way as a vector. However, the properties of complex numbers are very different from properties of vectors. The complex numbers form an entity which is called *field*. The real numbers are also a field. A field has to fulfill several axioms and the operations on complex numbers are defined in such a way that this is the case. We will not discuss the axiomatic approach further, but it is worth remembering, that the familiar properties of the field of real numbers are also satisfied by complex numbers. We will accept the following definitions for operations on complex numbers:
Sum:

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$$

Multiplication by a real number $a \in \mathcal{R}^1$:

$$az = (ax, ay)$$

These two oprations are analogous to operations on vectors. However, the product of two complex numbers has no analog in vector spaces:

$$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

From these definitions follows that the number $z = (0, 1)$ has an interesting property that its square is on the real axis at -1:

$$z^2 = zz = (0 - 1, 0 + 0) = (-1, 0)$$

or

$$z^2 = -1 \in \mathcal{R}^1$$

This special number is denoted by letter i, $i = (0, 1)$, and $i^2 = -1$. We sometimes say that i is the square root of -1. The definion of i allows one to write complex numbers in an alternative form

$$z = (x, y) = x + iy$$

This for is particularly useful for performing operations on complex numbers since such operations are now analogous to operations on polynomials. For example:

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + i^2 y_1 y_2 + ix_1 y_2 + iy_1 x_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + y_1 x_2)$$

in agreement with the definition given above.

We will now define the moduls or absolute value of a complex number and the complex conjugated numbers. For the former, we want by analogy to vectors to have

$$|z| = \sqrt{x^2 + y^2}$$

To express this quantitiy in complex arithmetic, define the complex number conjugated to z as

$$z^* = (x, -y) = x - iy$$

We have for the product

$$zz^* = (x + iy)(x - iy) = x^2 - i^2 y^2 + ixy - ixy = x^2 + y^2$$

so that we can write

$$|z| = \sqrt{zz^*}$$

Now we have to define functions of complex numbers. A direct definition is not possible for most functions, therefore we define such functions using their Taylor expansions. Since each Taylor expansion uses only additions and multiplications, and we know how to perform such operations on complex numbers, any function can be defined in this way. For example,

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$$

The functions of complex variables can be differentiated with respect to z, i.e., one can compute $df(z)/dz$. We will not need to define such a derivative in our course, but we will calculate a different, simpler type of derivatives of complex functions: with respect to a real parameter common to the real and imaginary parts of a complex number. Thus, if

$$z(t) = x(t) + iy(t)$$

then

$$\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$$

There are several useful relations that complex functions fulfill. One of them is the Euler relation. For a number $\theta \in \mathcal{R}^1$

$$\mathrm{e}^{i\theta} = \sum \frac{(i\theta)^n}{n!} = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots + i\left(\theta + \frac{\theta^3}{3!} + \cdots\right)$$

where we used the property $i^2 = -1$ and grouped real and imaginary components. We now recognize that the real component is an expansion of the function $\cos\theta$ and the imaginary one of $\sin\theta$, so that we have

$$\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$$

which is the Euler formula.

The Euler formula leads to the so-called "trigonometric" representation of complex numbers. Like in polar coordinates, for a number $z = (x, y)$, $x = r\cos\theta$ and $y = r\sin\theta$, where $r = |z|$ and $\theta = \arctan(y/x)$. Thus,

$$z = r\cos\theta + ir\sin\theta = r\mathrm{e}^{i\theta}$$

where we used Euler's formula in the last step. Since θ parametrizes z, we can calculate derivatives of z with respect to θ using the definition given above.