Classical Mechanics Lecture Notes
WORK AND ENERGY

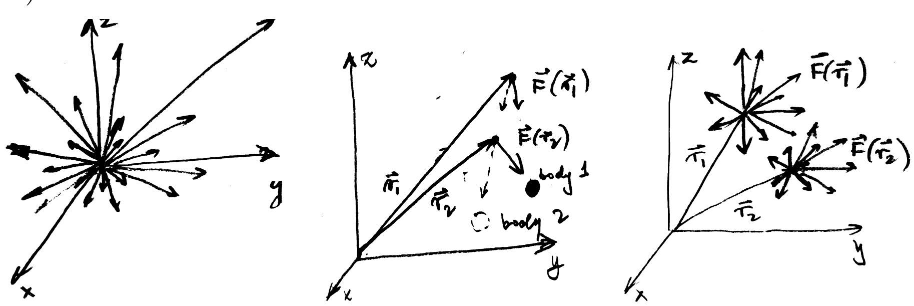
September 11, 2024

I. WORK AND ENERGY

We now want to introduce the concept of energy and derive the mechanical energy conservation theorem. Note the words *derive* and *theorem.* In contrast to some texts which present this subject as a new physical law, we will show that it is the consequence of Newton's laws. Thus, it is a theorem.

A. Force field

When we have discussed vectors, we stressed that in 3-dimensional space vectors are objects that can be determined by specifying exactly three components (three numbers). These numbers determine the direction, sense, and length of the vector, but not "the origination point" for a vector. Thus, all vectors with the same direction, sense, and length are equivalent. In other words, the origin of a vector is arbitrary and we can for simplicity assume that all vectors originate from the same point in space, for example the point [0,0,0]. Then the complete vector space is the set of all possible vectors originating from this point (see the figure).



While this definition is sufficient for many purposes, in physics one often encounters objects that are vectors but are connected with a given point in space. For example, the gravitational force that a body exerts on another body depends on the position in space where we measure this force, see the figure. The force at point $\boldsymbol{r}_1$ is different from the force at point $\boldsymbol{r}_2$. If instead of this body (body 1 in the figure), the field is due to another body (body 2), the force vectors are different (broken-line arrows in the figure). By changing the position and mass of the body, one can produce a force of arbitrary direction and magnitiude. Thus, at each point of the "position" space there is located a *separate* vector space. We differentiate between the vector spaces located at different $\boldsymbol{r}$'s by writing the vectors as $\boldsymbol{F}(\boldsymbol{r})$. Now we have six components in each object $\boldsymbol{F}(\boldsymbol{r})$, so we can have vectors assigned to different origins. Each component of the vector is a function of $\boldsymbol{r}$, e.g., $F_x = F_x(\boldsymbol{r}) = F_x(x, y, z)$. If we chose some value for $\boldsymbol{F}$ at each $\boldsymbol{r}$, we say that a *vector field* $\boldsymbol{F}(\boldsymbol{r})$ has been determined in space. If $\boldsymbol{F}$ is a force, we use the phrase *force field*. Other important vector fields in physics are the electric and magnetic fields. Note that the vector spaces at different $\boldsymbol{r}$ are separate vector spaces, so it does not make sense (and there is no need) to talk about the sums like $\boldsymbol{F}(\boldsymbol{r}_1) + \boldsymbol{F}(\boldsymbol{r}_2)$.

In a general case, the force field can depend on time and it will then be denoted as $\boldsymbol{F}(\boldsymbol{r}, t)$. The force may also depend on the velocity of the particle interacting with the field: $\boldsymbol{F}(\boldsymbol{r}, \boldsymbol{v}, t)$.

B. Work and energy on simple examples

In this subsection we will introduce the concepts of work and energy on simple one-dimensional examples. These concepts arise naturally upon integration of Newton's equation.

1. *Constant gravitational force field*

Consider vertical motion (with the y axis pointing up) of a body of mass m in Earth's gravitational field (near the surface of the Earth, so that the gravitational force is constant, $\boldsymbol{F} = -mg\hat{\boldsymbol{y}}$, where g is the free-fall acceleration). We can define work performed by the force on the body when the body moves from y_1 to y_2 as the integral of the force times displacement

$$W_{12} = \int_{y_1}^{y_2} F_y dy = \int_{y_1}^{y_2} (-mg) dy = -mg \int_{y_1}^{y_2} dy.$$

This integral can be evaluated to give

$$W_{12} = -mgy|_{y_1}^{y_2} = -mg(y_2 - y_1).$$

We see that the work integral defines a function, called potential energy,

$$U(y) = mgy$$

such that

$$F_y = -\frac{dU(y)}{dy}.$$

We can therefore write:

Definition: Gravitational potential energy of a particle in a constant gravitational field $-mg\hat{\boldsymbol{y}}$ is defined as

$$U \equiv U_g = mgy + C$$

where C is an arbitrary constant. The work integral can now be written as

$$W_{12} = -\left[U(y_2) - U(y_1)\right] = -(U_2 - U_1). \tag{1}$$

Thus, the work performed by the force on the body is equal the to minus the difference of potential energies. As an example, consider a body in free fall from $y_1 > y_2$. The work integral is positive since F_y and dy are both negative. Since $U_1 > U_2$, $\Delta U = U_2 - U_1$ is negative. As the body moves down, its potential energy decreases in magnitude by exactly the amount as the work performed on the body.

The work integral leads also to another kind of energy, called kinetic energy. To define this energy, plug in Newton's equation $ma_y = F_y$ into the work integral

$$\begin{aligned} W_{12} = \int_{y_1}^{y_2} F_y dy = \int_{y_1}^{y_2} ma_y dy = m\int_{y_1}^{y_2} \frac{dv_y}{dt} dy = m\int_{t_1}^{t_2} \frac{dv_y}{dt}\frac{dy}{dt} dt = \\ = m\int_{t_1}^{t_2} v_y \frac{dv_y}{dt} dt = m\int_{v_{y1}}^{v_{y2}} v_y dv_y. \end{aligned}$$

The second equality is just the definition of the acceleration and in the third one we used the relation for infinitesimal change of the dependent variable

$$dy = \frac{dy}{dt} dt.$$

Since we used Newton's equation, we know that its solution $y(t)$ relates points $y_i = y(t_i)$, which allowed use to change variables and limits of integration from displacement to time. Finally, we changed the variable one more time in a similar way using

$$dv_y = \frac{dv_y}{dt} dt,$$

with $v_{yi} = v_y(t_i)$. The last integral is elementary and we get

$$W_{12} = \frac{m}{2} v_y^2 \Big|_{v_{y1}}^{v_{y2}} = \frac{mv_{y2}^2}{2} - \frac{mv_{y1}^2}{2}.$$

The final quantity leads to a definition:

Definition: Kinetic energy of a particle is defined as

$$K = K(t) = \frac{mv_y^2}{2}.$$

We can also write

$$W_{12} = K(t_2) - K(t_1) = K_2 - K_1. \tag{2}$$

This equation is called the *work–kinetic energy theorem.* Thus, continuing with the example of falling body, the work performed on the body results in an increase of its kinetic energy.

Comparing Eqs. (1) and (2), we get

$$W_{12} = K_2 - K_1 = -(U_2 - U_1) \tag{3}$$

or

$$K_1 + U_1 = K_2 + U_2.$$

If we now define the total mechanical energy as

$$E \equiv E_{\text{mech}} = K + U,$$

we can write

$$E_{\text{mech}} = \text{const.}$$

which is the *mechanical energy conservation theorem*. Notice that this result was derived from Newton's equation using only mathematical transformations. Going back to our example of a falling body and looking at Eq. (3), we see that a positive work performed on a body by a force field decreases its potential energy while it increases its kinetic energy by the same amount.

2. *Spring force*

The arguments given above can be repeated for $F_x = -k(x - x_e) = -k\Delta x$, i.e., for the spring force. Let us choose the zero of the x axis such that $x_e = 0$. The relation (2) remains the same since in the derivation leading to the kinetic energy the actual form of the force is not used. However, the derivation leading to the potential energy has to performed again

$$W_{12} = \int_{x_1}^{x_2} F_x dx = \int_{x_1}^{x_2} (-kx) dx = -\frac{k}{2} x^2 \Big|_{x_1}^{x_2} = -\frac{k}{2}(x_2^2 - x_1^2) = -[U(x_2) - U(x_1)],$$

where

$$U_s(x) \equiv U(x) = \frac{1}{2} k x^2; \quad F_x = -\frac{dU(x)}{dx}.$$

Thus, the potential energy exists also for the spring force. Therefore, the mechanical energy is conserved also for this system.

C. Line integrals

The line integral is very generally defined as follows (Riemann's definition)

$$\int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \lim_{N\to\infty} \sum_{i=0}^{N-1} \boldsymbol{F}(\boldsymbol{r}_i) \cdot \Delta \boldsymbol{r}_i \tag{4}$$

i.e., we divide the path C into N segments $\Delta \boldsymbol{r}_i$, project $\boldsymbol{F}$ computed at the beginning of the segment (actually, at an arbitrary point on the segment) on $\Delta \boldsymbol{r}_i$ (via the dot product), and compute the sum of all such contributions. Whereas the Riemann definition gives us a good intuitive understanding of line integrals, and forms the basis of various numerical methods of integral calculations, it does not tell us how to compute line integrals analytically. Below we discuss analytic calculations, first for two special simple cases and then we derive a general parametrization method.

1. Straight line path

In this case, $|d\boldsymbol{r}| = |ds|$, where ds is an infinitesimal segment of the straight line. We therefore immediately get a simple one-dimensional integral

$$\int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{s_1}^{s_2} F(s) \cos\theta(s) ds$$

where θ is the angle between $\boldsymbol{F}$ and the line.

2. A path with linear segments parallel to the axes

If C consists of segments parallel to the axes of a coordinate system, the integral can be easily computed by using the component definition of dot product. Since $d\boldsymbol{r} = [dx, dy, dz]$, we have

$$\int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} [F_x dx + F_y dy + F_z dz].$$

Since on each line segment two out of three displacements $\{dx, dy, dz\}$ are zero, the integral becomes a sum of one-dimensional integrals.

3. Parametrization method

Each curve in two or three dimensions can be parametrized by a single parameter. For example, points on a circle of radius r are given by $x = r\cos\theta$ and $y = r\sin\theta$. We therefore have to first find a parametric form of the curve C, i.e., the three components of all the points on the curve should be given by functions dependent on a single parameter θ:

$$\boldsymbol{r}(\theta) = [x(\theta), y(\theta), z(\theta)].$$

The beginning and end points correspond then to the values of θ equal to θ_1 and θ_2, respectively, i.e., $\boldsymbol{r}_i = \boldsymbol{r}(\theta_i)$. We can now write

$$\int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{\theta_1}^{\theta_2} \boldsymbol{F}(\boldsymbol{r}(\theta)) \cdot \frac{d\boldsymbol{r}(\theta)}{d\theta} d\theta =$$

$$= \int_{\theta_1}^{\theta_2} \left[F_x(x(\theta), y(\theta), z(\theta))\, x'(\theta) + F_y(x(\theta), y(\theta), z(\theta))\, y'(\theta) + F_z(x(\theta), y(\theta), z(\theta))\, z'(\theta) \right] d\theta \tag{5}$$

where $x'(\theta) = dx(\theta)/d\theta$ so that the change of variables gives $dx = x'(\theta)d\theta$, and similarly for y and z variables. The last integral is just an integral of a scalar function of a single variable, so we know how to compute it.

Note that an often made mistake in calculating the line integrals is to use the second expression in Eq. (5) and treat each of the three terms appearing there as a one-dimensional integral (with appropriate assumptions about the values of the other two variables and the limits of integration). While in principle this procedure can lead to the correct answer, in practice it is very easy to make errors in this approach. The only case when such approach can be recommended is when the path of integration consists of linear segments parallel to the axes of the coordinate system.

Example:
Calculate the line integral for $\boldsymbol{F} = x^3 y \hat{\boldsymbol{x}} + (x - y)\hat{\boldsymbol{y}}$ along the parabola $y = x^2$ from $\boldsymbol{r}_1 = [-2, 4]$ to $\boldsymbol{r}_2 = [1, 1]$.

In the case of parabola, the parametrization is trivial: $x = \theta$ and $y = \theta^2$, with $\theta_1 = -2$ and $\theta_2 = 1$. We can write the force as $\boldsymbol{F} = \theta^5 \hat{\boldsymbol{x}} + (\theta - \theta^2)\hat{\boldsymbol{y}}$. Since $\boldsymbol{r}(\theta) = \theta\hat{\boldsymbol{x}} + \theta^2\hat{\boldsymbol{y}}$,

$$\frac{d\boldsymbol{r}}{d\theta} = \hat{\boldsymbol{x}} + 2\theta\hat{\boldsymbol{y}}$$

and we can write

$$\int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{\theta_1}^{\theta_2} \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{d\theta} d\theta = \int_{\theta_1}^{\theta_2} [\theta^5 + 2(\theta - \theta^2)\theta] d\theta = \left[\frac{1}{6}\theta^6 + \frac{2}{3}\theta^3 - \frac{1}{2}\theta^4 \right] \Bigg|_{-2}^{1} = 3.$$

D. General definition of work

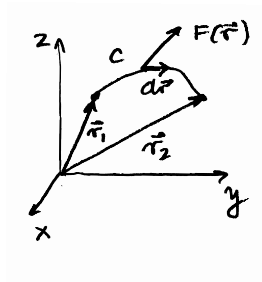
From everyday experience, work is connected with the magnitude of the force acting during the process of performing the work and the magnitude of the displacement of a body. Thus, the following definition should appear to be "natural". We first define an infinitesimal work by the force $\boldsymbol{F}(\boldsymbol{r})$ displacing the body from $\boldsymbol{r}$ to $\boldsymbol{r} + d\boldsymbol{r}$ as

$$dW = \boldsymbol{F} \cdot d\boldsymbol{r}. \tag{6}$$

Now assume that a body is moved from point $\boldsymbol{r}_1$ to point $\boldsymbol{r}_2$ along path C, as shown on the figure. Then the work done by the force field $\boldsymbol{F}$ on the body is the following line integral

$$W_{12} = \int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}. \tag{7}$$

In general, the work does depend on the path, so the path has to be specified. The definition of work can be used both for the net force acting on the body (i.e., the sum of all forces) and for each of the particular forces acting on this body.



E. Conservative force field

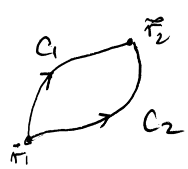
We define *conservative* force field as the field having the property that the integral (7) is independent of the path C.

Definition: A field is conservative if and only if

$$\int_{r_1:C_1}^{r_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{r_1:C_2}^{r_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} \tag{8}$$

for any two paths C_i.

This theorem is illustrated on the figure below. The reason for using the name "conservative" will be given later.



We will now prove the following auxiliary theorem

Theorem 1: The force $\boldsymbol{F}(\boldsymbol{r})$ is conservative $\Leftrightarrow \oint \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = 0$. The integral with the circle symbol denotes a closed-loop line integral.

Proof ($\Rightarrow$):

$$\begin{aligned}
\oint \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_{r_1:C_1}^{r_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} + \int_{r_2:C_2}^{r_1} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} \\
&= \int_{r_1:C_1}^{r_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} - \int_{r_1:C_2}^{r_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = 0
\end{aligned} \tag{9}$$

since the field was assumed to be conservative.

Proof ($\Leftarrow$):

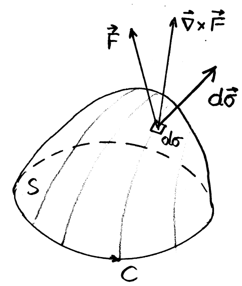
Consider the figure below. We have two closed path integrals, one over the path $C_1 + C_2$ and another one over $C_1 + C_2'$. Each integral is zero by assumption. Since the integrals have a common part over C_1, the integral over C_2 must be equal to the integral over C_2'.

1. Curl of $\boldsymbol{F}$ for conservative force fields

There exists another useful relation for checking if a force field is conservative. It utilizes the familiar Stokes theorem

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{\sigma} = \oint_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} \tag{10}$$

where the first integral is over the surface S and the second integral is over the curve C bounding S. The differential $d\boldsymbol{\sigma}$ is a vector perpendicular to the surface S at a given point



and having the magnitude of the surface area element, see the figure. Recall that the curl of a vector is defined as:

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

We can now formulate the following theorem, allowing one to use $\boldsymbol{\nabla} \times \boldsymbol{F}$ to check if a field $\boldsymbol{F}$ is conservative

Theorem 2:

$$\boldsymbol{\nabla} \times \boldsymbol{F} = 0 \quad \Leftrightarrow \quad \oint_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = 0 \quad (\Leftrightarrow \quad \boldsymbol{F} \textit{ is conservative})$$

where C is an arbitrary closed contour in some region of space. The proof of this theorem results immediately from the Stokes theorem. In the $\Rightarrow$ direction, $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$ implies that the integral on the left-hand side of Eq. (10) is zero, so the integral on the right-hand side, which is the same as the integral appearing in Theorem 2, has to be zero as well. In the $\Leftarrow$ direction, we can choose as C any contour in space. Then the surface integral analogous to that on the left-hand side of Eq. (10), but taken over any surface S bound by C, will be zero. Thus, such integral over an arbitrary surface will be zero. This is possible only if the integrand is zero. [If this is not yet convincing, realize that if the integral on the left-hand side of Eq. (10) (over the complete S) is zero as implied by the assumption, and the integrand is not zero, this means that there are positive and negative contribution to this integral which cancel each other. Then consider the subarea which gives a positive contribution (such that the integrand is positive everywhere). This subarea is bounded by a contour and the line integral over this contour is zero by assumption. Therefore, the surface integral has also to be zero and cannot, in fact, be positive. This implies again that $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$].

F. Potential energy for conservative force field

One can easily generalize the findings of Sec. I B

$$\text{gravity} \quad W_{12} = \int_{y_1}^{y_2} F_y dy = \int_{y_1}^{y_2} (-mg) dy = U(y_1) - U(y_2); \quad U(y) = mgy; \quad F_y = -\frac{dU(y)}{dy}$$

$$\text{spring} \quad W_{12} = \int_{x_1}^{x_2} F_x dx = \int_{x_1}^{x_2} (-kx) dx = U(x_1) - U(x_2); \quad U(x) = \frac{1}{2}kx^2; \quad F_x = -\frac{dU(x)}{dx}$$

to a general one-dimensional case and is just given by the derivative-antiderivative relations. If

$$\int_{x_1}^{x_2} F(x) dx = G(x_2) - G(x_1)$$

where the function $G(x)$ is the antiderivative of the function $F(x)$

$$F(x) = \frac{dG}{dx}.$$

Then the potential energy is just $U(x) = -G(x)$. We do not need any special proof of this relation since it is a calculus theorem. However, where does the concept of conservative force enters? In one dimension, there is only one path from x_1 to x_2, so can conservative force be defined? Yes, by considering a closed path, i.e., $x_1 \to x_2 \to x_1$. If F_x is given by a single function, the integral over this closed path is zero from the change of limits changes the sign of the integral theorem. This works therefore for both gravity and spring forces. If we have to use a different definition of the force on the two segments, for example, the force of friction will be negative on the path $x_1 \to x_2$ and positive on the path $x_2 \to x_1$, the theorem will not hold.

The three-dimensional case is more difficult. For a conservative force $\boldsymbol{F}$, the work integral depends only on the initial and final points on the path

$$\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{r}') \cdot d\boldsymbol{r}' = H_{\boldsymbol{r}_0}(\boldsymbol{r}) \tag{11}$$

(note that we omitted "C" from the integral symbol since this integral is independent of C). If we keep $\boldsymbol{r}_0$ constant, the work integral defines a unique function in the whole space. For some other choice of a fixed initial point, $\boldsymbol{r}_0'$, the new function differs only by a constant, equal to the integral from $\boldsymbol{r}_0'$ to $\boldsymbol{r}_0$ (see the figure below). Thus, a negative of $H_{\boldsymbol{r}_0}(\boldsymbol{r})$,

$$U(\boldsymbol{r}) + C = -H(\boldsymbol{r}) + C = -H_{\boldsymbol{r}_0}(\boldsymbol{r}),$$

has features of the potential energy. [The functions $U(\boldsymbol{r}) = -H(\boldsymbol{r})$ are determined up to an arbitrary additive constants and therefore do not have subscript $\boldsymbol{r}_0$]. What remains to be shown is that the negative of the gradient of U gives the force.

The gradient of U is denoted as $\boldsymbol{\nabla} U$ (pronounced also "del U" or "nabla U") is defined as:

$$\boldsymbol{\nabla} U(x,y,z) = \left[\frac{\partial U(x,y,z)}{\partial x}, \frac{\partial U(x,y,z)}{\partial y}, \frac{\partial U(x,y,z)}{\partial z}\right] = \frac{\partial U}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial U}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial U}{\partial z}\hat{\boldsymbol{z}}.$$

where the partial derivatives of $U(x,y,z)$ are the derivatives with respect to one variable with the other two variables kept constant. For example,

$$\frac{\partial U(x,y,z)}{\partial x} = \frac{d}{dx} U(x,y,z)\Bigg|_{y,z=\text{const.}}.$$

If the variables are incremented by infinitesimal amounts dx, dy, and dz, the infinitesimal change of U is the sum changes along the three directions

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz.$$

It has to be so since other contributions to the change of U would have to depend on products such as, for example, $dx\,dy$ which go to zero faster than dx or dy. This relation can be written in a compact form as

$$dU = \boldsymbol{\nabla} U \cdot d\boldsymbol{r}. \tag{12}$$

To find the relation between U and $\mathbf{F}$ in the simplest way, let us consider Eq. (11) for the special case where $\boldsymbol{r}_0$ and $\boldsymbol{r}$ are so close to each other that their difference is an infinitesimal increment $d\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_0$. Then the integral on the left-hand side of Eq. (11) is just equal to $\boldsymbol{F} \cdot d\boldsymbol{r}$, according to Riemann's definition, whereas the right-hand side is $-dU$ (assuming $U(\boldsymbol{r}_0) = 0$)

$$\boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = -dU$$

Comparing this with Eq. (12), we can write

$$\boldsymbol{F} = -\boldsymbol{\nabla} U.$$

To prove these relations in a more rigorous way, let us write the theorem we want to prove

Theorem 3: If the force $\boldsymbol{F}(\boldsymbol{r})$ is conservative $\Leftrightarrow$ there exists a function $H(\boldsymbol{r})$ such that $\boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{\nabla} H(\boldsymbol{r})$.

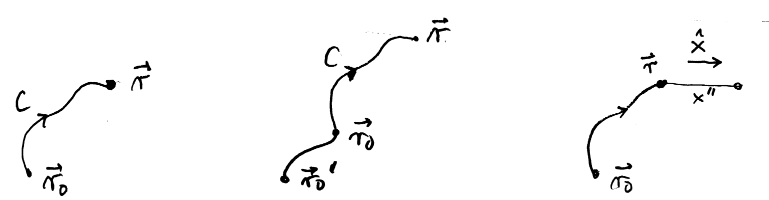
Proof ($\Rightarrow$):
Consider the integral from $\boldsymbol{r}_0$ to $\boldsymbol{r}$ over path C. Due to definition of conservative force, this integral is independent of the choice of the path. [For easy reading, we repeat some material covered above]. Thus, for some fixed $\boldsymbol{r}_0$, the integral

$$\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{r'}) \cdot d\boldsymbol{r'}$$

(there is no C in the symbol of integral to stress the independence of path) gives always the same value for a given $\boldsymbol{r}$. Therefore, as we change $\boldsymbol{r}$, the integral defines, as already discussed, a *unique* function of $\boldsymbol{r}$:

$$\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{r'}) \cdot d\boldsymbol{r'} = H_{\boldsymbol{r}_0}(\boldsymbol{r}). \tag{13}$$

Notice that if we choose another starting point, for example $\boldsymbol{r}_0'$ marked on the figure below, the new function $H_{\boldsymbol{r}_0'}(\boldsymbol{r})$ differs from $H_{\boldsymbol{r}_0}(\boldsymbol{r})$ only by a constant, equal to the value of the integral from $\boldsymbol{r}_0'$ to $\boldsymbol{r}_0$. So far we restated the arguments made at the beginning of this section.



Now consider a point in space which is shifted with respect to $\boldsymbol{r}$ by x'' along the $\hat{\boldsymbol{x}}$ axis (see the figure above, the right panel). Note the x'' is not infinitesimal. The value of $H_{\boldsymbol{r}_0}$ at this point is

$$H_{\boldsymbol{r}_0}(x + x'', y, z) = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}+x''\hat{\boldsymbol{x}}} \boldsymbol{F}(\boldsymbol{r'}) \cdot d\boldsymbol{r'} = H_{\boldsymbol{r}_0}(x, y, z) + \int_{\boldsymbol{r}}^{\boldsymbol{r}+x''\hat{\boldsymbol{x}}} \boldsymbol{F}(\boldsymbol{r'}) \cdot d\boldsymbol{r'}. \tag{14}$$

The last integral in the equation above is just a one-dimensional integral since on this path $d\boldsymbol{r'} = dx'\hat{\boldsymbol{x}}$, i.e., $dy = dz = 0$. Therefore, we can write

$$\int_{\boldsymbol{r}}^{\boldsymbol{r}+x''\hat{\boldsymbol{x}}} \boldsymbol{F}(\boldsymbol{r'}) \cdot d\boldsymbol{r'} = \int_{\boldsymbol{r}}^{\boldsymbol{r}+x''\hat{\boldsymbol{x}}} F_x(x', y, z) dx' \tag{15}$$

where F_x denotes the x component of $\boldsymbol{F}$. In this integral, the values of y' and z' are fixed and equal to y and z, respectively, as utilized in writing the integrand. Thus, the integral written above is equivalent to the standard integral of a single variable $\int g(x')dx'$ with $g(x) = F_x(x, y, z)|_{y,z=\text{const.}}$. Denoting by $G(x)$ the antiderivative of $g(x)$, i.e.,

$$F_x(x, y, z)|_{y,z=\text{const.}} = g(x) = \frac{dG(x)}{dx},$$

we can use the definition of the integral:

$$\int_{\boldsymbol{r}}^{\boldsymbol{r}+x''\hat{\boldsymbol{x}}} F_x(x', y, z)dx = \int_x^{x+x''} g(x')dx' = \int_x^{x+x''} \frac{dG(x')}{dx'}dx' = G(x + x'') - G(x). \qquad (16)$$

Now, comparing this to Eqs. (14) and (15), we see that

$$\int_{\boldsymbol{r}}^{\boldsymbol{r}+x''\hat{\boldsymbol{x}}} F_x(x', y, z)dx' = H_{\boldsymbol{r}_0}(\boldsymbol{r} + x''\hat{\boldsymbol{x}}) - H_{\boldsymbol{r}_0}(\boldsymbol{r}) = G(x + x'') - G(x),$$

where the first equality is just reshuffled Eq. (14) with the integral replaced by that of Eq. (15) and the second one is Eq. (16). Therefore, G and $H_{\boldsymbol{r}_0}$ can differ only by a constant and we can write

$$F_x = \frac{\partial H_{\boldsymbol{r}_0}}{\partial x}. \qquad (17)$$

Similar relations can be obtained for F_y and F_z by considering appropriate displacements from $\boldsymbol{r}$. We therefore can write:

$$\boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{\nabla} H_{\boldsymbol{r}_0}(\boldsymbol{r}) \qquad (18)$$

and the function $H(\boldsymbol{r})$ in the theorem is equal to $H_{\boldsymbol{r}_0}(\boldsymbol{r})$ plus an arbitrary constant

$$H(\boldsymbol{r}) = H_{\boldsymbol{r}_0}(\boldsymbol{r}) + \text{const.}$$

The arguments leading to Eq. (17) can be simplified by making x'' infinitesimal: $x'' \to dx''$. Then Eq. (15) reduces to $F_x(x, y, z)dx''$ and

$$H_{\boldsymbol{r}_0}(\boldsymbol{r} + dx''\hat{\boldsymbol{x}}) - H_{\boldsymbol{r}_0}(\boldsymbol{r}) = \frac{\partial H_{\boldsymbol{r}_0}}{\partial x}dx'',$$

leading to

$$F_x(x, y, z)dx'' = \frac{\partial H_{\boldsymbol{r}_0}}{\partial x}dx'',$$

which gives Eq. (17). This reasoning is similar to the one at the beginning of this section.

Proof ($\Leftarrow$):
If $\boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{\nabla} H(\boldsymbol{r})$ then

$$\oint \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \oint \boldsymbol{\nabla} H(\boldsymbol{r}) \cdot d\boldsymbol{r} = \oint dH = H(\boldsymbol{r}_0) - H(\boldsymbol{r}_0) = 0 \qquad (19)$$

where we have used

$$dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy + \frac{\partial H}{\partial z}dz = \boldsymbol{\nabla} H(\boldsymbol{r}) \cdot d\boldsymbol{r}.$$

This completes the proof. The intermediate step in (19) can be understood better by writing explicitly the definition of the line integral

$$\oint \boldsymbol{\nabla} H(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{t_1}^{t_2} \boldsymbol{\nabla}\phi(\boldsymbol{r}) \cdot \frac{d\boldsymbol{r}}{dt} dt = \int_{t_1}^{t_2} \left[\frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial z}\frac{dz}{dt}\right] dt = \int_{t_1}^{t_2} \frac{dH}{dt} dt.$$

One may remark here that the equation we have proved above

$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{\nabla} H(\boldsymbol{r}) \cdot d\boldsymbol{r} = H(\boldsymbol{r}_2) - H(\boldsymbol{r}_1)$$

is the three-dimensional equivalent of the familiar formula

$$\int_{x_1}^{x_2} \frac{dG(x)}{dx} dx = G(x_2) - G(x_1).$$

Furthermore, in future work we will often see Eq. (13) written as:

$$H(\boldsymbol{r}) = H(\boldsymbol{r}_0) + \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{r'}) \cdot d\boldsymbol{r'}.$$

In physics, the function H is traditionally taken with sign reversed, denoted by the letter U, and called the potential energy, so that we have

$$\boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{\nabla} U(\boldsymbol{r}). \tag{20}$$

We can therefore write the relation between work and the potential in one more form

$$-[U(\boldsymbol{r}_2) - U(\boldsymbol{r}_1)] = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}. \tag{21}$$

We can now formulate the final form of Theorem 3 (split into two theorems)

Theorem 3a: *For any conservative force field $\boldsymbol{F}(\boldsymbol{r})$ there exists a scalar function $U(\boldsymbol{r})$ such that $\boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{\nabla} U(\boldsymbol{r})$. The function $U(\boldsymbol{r})$ is called the potential energy.*

As it is obvious from the theorems discussed earlier, the inverse theorem is also true, i.e.,

Theorem 3b: *If a force field can be written as $\boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{\nabla} U(\boldsymbol{r})$, then this force field is conservative.*

One should point out that the whole reasoning in the present section (I F) is purely mathematical and involves no physics (including no physics assumptions). This reasoning is valid for an arbitrary vector field.

G. Work–kinetic energy theorem

As some forces act on a body, and the net force is nonzero, the body accelerates and the magnitiude of the velocity is changing. We know from experience that depending on this magnitude and the mass of the body, the effects of a collision of the body with another object can be quite different. One quantity which depends on velocity and mass is the momentum, a vector quantity. Kinetic energy is a scalar quantity which will reflect this changing state of a body.

Consider again a body of mass m which is initially, at time t_1, at $\boldsymbol{r}_1 = \boldsymbol{r}(t_1)$ and it arrives at time t_2 at $\boldsymbol{r}_2 = \boldsymbol{r}(t_2)$. The motion is along a path C. The body is accelerated by a net force $\boldsymbol{F}(\boldsymbol{r})$ acting on it. Let us try to relate the work W_{12} to the initial and final velocities of the body, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Using Newton's equation, $\boldsymbol{F} = m\boldsymbol{a}$, we can write Eq. (7) as

$$\begin{aligned} W_{12} &= \int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = m \int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} \boldsymbol{a} \cdot d\boldsymbol{r} = m \int_{t_1}^{t_2} \left(\frac{d\boldsymbol{v}}{dt}\right) \cdot \left(\frac{d\boldsymbol{r}}{dt} dt\right) \\ &= m \int_{t_1}^{t_2} \left(\frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v}\right) dt = \frac{m}{2} \int_{t_1}^{t_2} \frac{d(\boldsymbol{v} \cdot \boldsymbol{v})}{dt} dt = \frac{m}{2} \int_{t_1}^{t_2} \frac{dv^2}{dt} dt \\ &= \frac{m}{2} v^2 \Big|_{t_1}^{t_2} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = K_2 - K_1 \end{aligned} \tag{22}$$

where to get the parametric form of the work integral we have chosen time t as the parameter and we have used the definition of kinetic energy

$$K = \frac{mv^2}{2}, \tag{23}$$

the same definition as introduced in the one-dimensional case. We have used in the derivation the identity

$$\frac{d}{dt}(\boldsymbol{v} \cdot \boldsymbol{v}) = \frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \frac{d\boldsymbol{v}}{dt} = 2(\frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v}).$$

We have also used the definition of the integral

$$\int \frac{df(x)}{dx} dx = f(x) + C.$$

Thus, the work performed on a particle by the net force is equal to the change of the kinetic energy of this particle:

$$W_{12} = K_2 - K_1 = \Delta K \tag{24}$$

This equation is the "work–kinetic energy" theorem now derived for a general case of motion in three dimensions. Note that this theorem applies to both conservative and nonconservative forces. Also note that although the derivation is purely mathematical, we used Newton's equation, so the theorem is a part of physics.

H. Mechanical energy conservation theorem

If a force field is conservative, we can combine Eqs. (7), (24), and (21) together

$$W_{12} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = -[U(\boldsymbol{r}_2) - U(\boldsymbol{r}_1)] = -[U_2 - U_1] = K_2 - K_1 \tag{25}$$

which implies

$$K_1 + U_1 = K_2 + U_2.$$

Since the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are arbitrary, it follows that

$$T + U = E \equiv E_{\text{mech}} = \text{const.} \tag{26}$$

where T and U are the kinetic and potential energies, respectively, at some arbitrary point and we have denoted their sum by E. This sum will be called the (total) mechanical energy of the system. Equation (26) is the mechanical energy conservation theorem. Note that this equation is rigorously derived from Newton's equation (via the work–kinetic energy theorem), therefore it should be called a theorem, not a law. Equations (25) and (26) also imply useful relations $dW = -dU = dT$ and $dT + dU = 0$.

All the derivations so far assumed that U is only a function of $\boldsymbol{r}$. Can it be also an explicit function of time: $U = U(\boldsymbol{r}, t)$? All the derivations until that of the present section remain correct if the time is kept constant. Therefore,

$$\boldsymbol{F}(\boldsymbol{r}, t) = -\boldsymbol{\nabla} U(\boldsymbol{r}, t).$$

The derivation of Sec. I G does not depend on U. However, Eq. (25) is not true anymore. This is because the first equality is true only on the path determined by Newton's equation (it was used in Sec. I G), therefore $\boldsymbol{F}$ in this equation depends both on the position of the particle at time t and on this time explicitly: $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}(t), t)$. This makes the second equality in Eq. (25) invalid since it was proved only for constant t. Since a potential energy dependent explicitly on time does not lead to energy conservation theorem, it is not used in Newtonian mechanics and we instead use $\boldsymbol{F}(\boldsymbol{r}, t)$ directly in Newton's equations. However, in Lagrange's mechanics, which is formulated in terms of T and U, we assume that in general $U = U(\boldsymbol{r}, t)$.

I. Energy conservation law

As stressed before, the force in the work integral entering the work–kinetic energy theorem of Eq. (24) is the net force acting on a body. This force, in general, will have both the conservative and non-conservative components

$$\boldsymbol{F} = \boldsymbol{F}_{\text{net}} = \boldsymbol{F}_{\text{cons}} + \boldsymbol{F}_{\text{non-cons}}.$$

It is usually convenient to continue using the potential energy concept for the conservative part, leading to the following form of Eq. (24):

$$\Delta K = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}_{\text{cons}}(\boldsymbol{r}) \cdot d\boldsymbol{r} + \int_{\boldsymbol{r}_1:C}^{\boldsymbol{r}_2} \boldsymbol{F}_{\text{non-cons}}(\boldsymbol{r}) \cdot d\boldsymbol{r} = -[U_2 - U_1] + W_{12}^{\text{diss}}$$

where we denoted the work by the non-conservative force as $W_{12}^{\rm diss}$ for "dissipative". We will also sometimes use the term "dissipative" for the non-conservative forces. We can now write

$$\Delta E_{\rm mech} = \Delta K + \Delta U = W_{12}^{\rm diss}.$$

Thus, the mechanical energy is not conserved in the presence of dissipative forces. However, if we extend the energy concept to include the thermal energy, defined here as

$$\Delta E_{\rm th} = -W_{12}^{\rm diss},$$

it is possible to formulate the *energy conservation law*

$$\Delta E_{\rm tot} = \Delta E_{\rm mech} + \Delta E_{\rm th} = 0. \qquad (27)$$

This law is based on measurements of mechanical energy, which we know how to perform, and of thermal energy, which we do not know how to do, but this will be the subject of future courses. Equation (27) cannot be derived, it is a fundamental law of nature based on observations.

The microscopic base of Eq. (27) in the case of friction forces is very straightforward. As friction slows down the body, it achieves it by atomic-level forces between the surface and the body. These forces act on the atoms as well, accelerating their motions. The thermal energy is actually the due to the motions of these atoms (mostly vibrational for solid bodies).