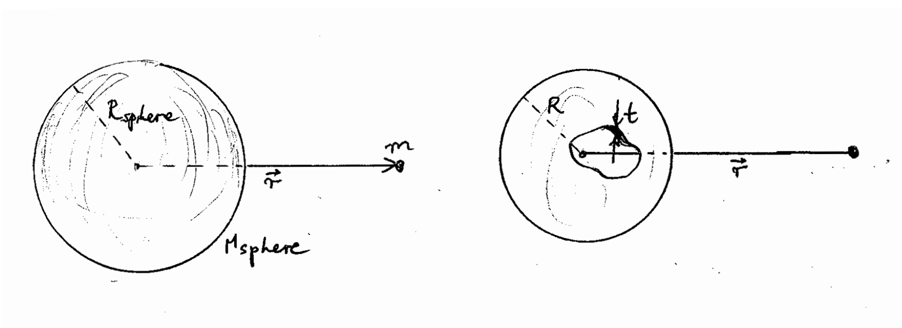
Classical Mechanics Lecture Notes

I. INTERACTION BETWEEN A SPHERE AND A POINT MASS

Newton's law of gravity is formulated for point masses. If two masses are very far apart, this law is obviously a very good approximation, independently of the shape of each body, but it clearly cannot be true for short separations between the two objects. However, if one of these bodies is a sphere of uniform density, the law holds even for arbitrarily small separations, i.e., the two bodies interact as if the sphere were a point placed in the center of the sphere with the mass equal to the mass of the sphere. Thus, the force on a point mass m due to the interaction between this mass and a sphere of mass M_{sphere}, with the separation between the center of the sphere and the point mass equal to r, is given by

$$\boldsymbol{F}_{\text{sphere}} = G\frac{M_{\text{sphere}}m}{r^2}(-\hat{\boldsymbol{r}}),$$

as illustrated on the figure below.

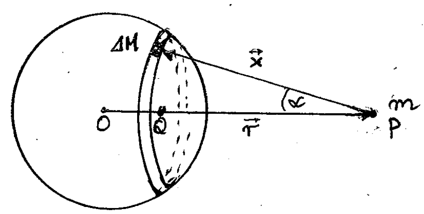


As the first step in proving this theorem, let us partition the sphere into spherical shells of radius $R < R_{\text{sphere}}$ and thickness t (think of layers of onion), as shown above. We want to prove that the shell exerts the force

$$\boldsymbol{F}_{\text{sphell}} = G\frac{M_{\text{sphell}}\, m}{r^2}(-\hat{\boldsymbol{r}})$$

on mass m. If we can prove this theorem, the theorem for the sphere follows immediately since the sum of masses of the shells is the mass of the sphere. Furthermore, we can then see that the sphere does not need to be of a uniform density, only the each sphere has to have a uniform density. This is why the theorem works for Earth whose density varies with the distance from the center.

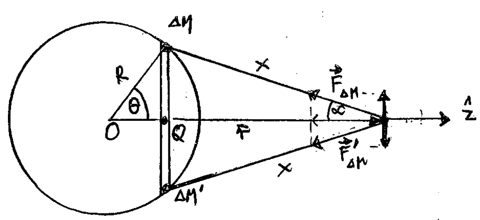
We will prove the theorem for the shell by dividing the shell into rings of infinitesimal width concentric with the vector $\boldsymbol{r}$, as shown in the figure on the next page. The distance from the mass m to any point on the ring is x since lines connecting this mass with points on the ring form a symmetric cone. Consequently, also the angle between any such line and vector $\boldsymbol{r}$ is the same for all lines and we will denote it by α.



Now divide the ring into small segments of mass ΔM such that the sum of all such masses is equal to the mass of the ring M_{ring}. For the interaction between masses ΔM and m we can use Newton's law of gravity for point masses

$$\boldsymbol{F}_{\Delta M} = G\frac{\Delta M\, m}{x^2}\hat{\boldsymbol{x}} \tag{1}$$

(note the orientation of vector $\boldsymbol{x}$). We see that the forces acting on the mass m from each element ΔM are the same in magnitude but differ in direction. Let us now consider two elements placed symmetrically on the ring, opposite to each other. In the plane formed by these elements shown on the figure below, the forces are symmetric and therefore the components perpendicular to $\boldsymbol{r}$ cancel and the components parallel to $\boldsymbol{r}$ are all equal to $-\left|\boldsymbol{F}_{\Delta M}\right|\cos\alpha$.



Thus, if we sum Eq. (1) over all ring elements, the masses ΔM will just sum to the mass of the ring and the total force from the ring on mass m will be

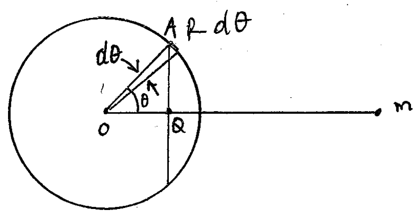
$$\boldsymbol{F}_{\text{ring}} = G\frac{M_{\text{ring}}\, m}{x^2}\cos\alpha(-\hat{\boldsymbol{r}}). \tag{2}$$

The only task remaining is to perform the sum (or rather integral) over the rings. First, let us express the position and width of each ring as a function of the spherical angle θ. We can see from the figure on the next page that the width of the disk is $Rd\theta$ and its radius is $AQ = R\sin\theta$. Therefore the volume of the ring is

$$V_{\text{ring}} = 2\pi t R\sin\theta R d\theta = 2\pi t R^2 \sin\theta d\theta$$

and its mass

$$M_{\text{ring}} = 2\pi\rho\, t R^2 \sin\theta d\theta,$$



where ρ is shell's density. If we plug this expression into Eq. (2), we obtain

$$\boldsymbol{F}_{\text{ring}} = G\frac{2\pi\rho\, tR^2 m}{x^2}\cos\alpha\sin\theta d\theta(-\hat{\boldsymbol{r}}). \tag{3}$$

To get the sum of contribution from all rings together, we can perform integration over θ after expressing x and α as functions of θ. It turns out that the integration is easier if it is performed over x, as it will be done below.

To express θ and α as functions of x, consider the triangle in the figure below. We have there

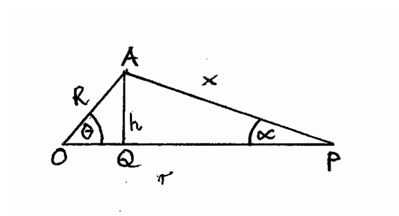
$$QP = x\cos\alpha \quad \text{and} \quad QP = r - R\cos\theta.$$

Also, from cosine theorem, we have

$$x^2 = r^2 + R^2 - 2rR\cos\theta \quad \Rightarrow \quad \cos\theta = \frac{r^2 + R^2 - x^2}{2rR}$$

so that

$$\cos\alpha = \frac{r}{x} - \frac{r^2 + R^2 - x^2}{2rx} = \frac{r^2 - R^2 + x^2}{2rx}.$$



Differentiating the expression for $\cos\theta$ over x

$$\frac{d\cos\theta}{dx} = \frac{d\cos\theta}{d\theta}\frac{d\theta}{dx} = -\frac{x}{rR}$$

we get

$$\sin\theta d\theta = \frac{x}{rR}dx.$$

Plugging this expression and the expression for $\cos\alpha$ into Eq. (3) we get

$$F_{\text{ring}} = \frac{\pi G\rho\, tRm}{r^2}\frac{r^2 - R^2 + x^2}{x^2}dx. \tag{4}$$

We can now integrate over dx, i.e., sum the contributions from all rings to get the magnitude of the force on the shell

$$F_{\text{shell}} = \int F_{\text{ring}} = C \int_{r-R}^{r+R} \frac{r^2 - R^2 + x^2}{x^2} dx$$

where C denotes the constant fraction in the previous equation and the limits of integration are consistent with the direction of the vector $\boldsymbol{x}$ marked on an earlier figure. The integral is very simple and we get

$$\begin{aligned}
F_{\text{shell}} = C(r^2 - R^2) \int_{r-R}^{r+R} \frac{1}{x^2} dx + C \int_{r-R}^{r+R} dx &= -C(r^2 - R^2) \frac{1}{x}\Big|_{r-R}^{r+R} + Cx\Big|_{r-R}^{r+R} \\
&= -C(r^2 - R^2)\left(\frac{1}{r+R} - \frac{1}{r-R}\right) + C(r + R - r + R) \\
&= -C(r^2 - R^2)\frac{r - R - r - R}{r^2 - R^2} + 2CR = 4CR
\end{aligned}$$

so that the force on the shell becomes

$$F_{\text{shell}} = \frac{\pi G \rho\, t R m}{r^2} 4R = \frac{4\pi G \rho\, t R^2 m}{r^2}$$

Since the volume of the shell is $4\pi R^2 t$ and therefore its mass is $4\pi R^2 t\rho$, we finally get

$$F_{\text{shell}} = G\frac{M_{\text{shell}}\, m}{r^2}$$

and indeed the shell interacts with a point mass m as if it were a point mass M_{shell} placed at the center of the shell.