Classical Mechanics Lecture Notes
NEWTON's LAW

September 6, 2024

I. INTRODUCTION

Classical mechanics is the most axiomatic branch of physics. It is based on only a very few fundamental concepts (quantities, "primitive" terms) that cannot be defined and virtually just one law, i.e., one statement based on experimental observations. All the subsequent development uses only logical reasoning. Thus, classical mechanics is nowadays sometimes considered to be a part of mathematics and indeed some research in this field is conducted at departments of mathematics. However, the outcome of the classical mechanics developments are predictions which can be verified by performing observations and measurements. One spectacular example can be predictions of solar eclipses for virtually any time in the future.

Various textbooks introduce different numbers of fundamental concepts. We will use the minimal possible set, just the concepts of space and time. These two are well-known from everyday experience. We live in a three-dimensional space and have a clear perception of passing of time. We can also easily agree on how to measure these quantities.

We will denote the position of a point in space by a vector $\boldsymbol{r}$. If we define an arbitrary Cartesian coordinate system, this vector can be described by a set of three components:

$$\boldsymbol{r} = [x, y, z] = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ are unit vectors along the axes of the coordinate system. If the point is moving in the coordinate system, $\boldsymbol{r} = \boldsymbol{r}(t)$. Thus, each component of $\boldsymbol{r}$ is a single-variable function, e.g., $x = x(t)$. The fundamental concepts of time and space allow us to define the velocity $\boldsymbol{v}$ and the acceleration $\boldsymbol{a}$ of the point in the standard way known from mathematics:

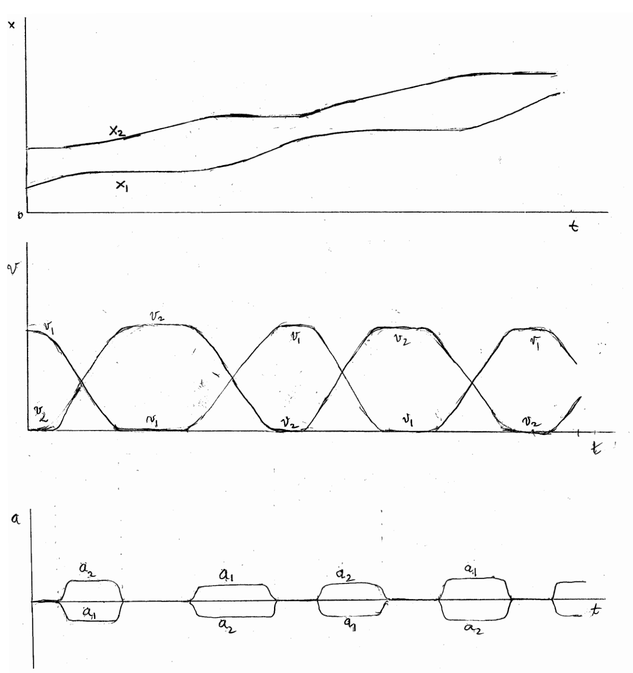
$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}(t)}{dt} = \left[\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt}\right]$$

$$\boldsymbol{a} = \ddot{\boldsymbol{r}} = \frac{d^2\boldsymbol{r}}{dt^2} = \frac{d\boldsymbol{v}}{dt}.$$

II. NEWTON'S LAW

With these definitions, we are ready to formulate Newton's law. Historically, there were three Newton's laws developed, and this approach is still followed by many textbooks. We will relate to these laws later on. To formulate Newton's law, we have to perform experiments. We will assume that the experiments are performed on two isolated bodies. This approximation can be well realized in practice with modern technology. We can make measurements on balls moving on nearly frictionless surfaces or motions of cars on air cushions. Our laboratory can also be the Universe and we can use telescopes to measure the motions of planets and stars. We measure only the positions of the objects as functions of time. From these data, we can calculate the velocity and acceleration of each body at each instant of time.

In particular, we can analyze the air-track experiment shown on the movie that can be found at: `http://www.youtube.com/watch?v=amfw2nABke4`. Notice that we have no information about this experiment other than we can see, so we can only find the positions of the objects as functions of time. If we analyze this motion frame by frame, we get the graphs shown below:



As we can see, the accelerations are strikingly correlated:

$$a_2 = -a_1 \quad \text{or} \quad \frac{|a_2|}{|a_1|} = 1 \quad \text{and} \quad a_1 \quad \text{opposite} \quad a_2$$

everywhere. This relation is particularly simple due to the fact that the two carts in the experiment are identical. If the two cars were different, we would have instead

$$\frac{|a_2|}{|a_1|} = k_{21} \quad \text{and} \quad a_1 \quad \text{opposite} \quad a_2$$

where k_{21} is a constant, always the same for two given carts.

The results of these and similar experiments can be summarized in the following law (generalizing to three dimensions and using the notation $\hat{\boldsymbol{a}} = \boldsymbol{a}/|\boldsymbol{a}|$, where $|\boldsymbol{a}|$ is the magnitude of vector $\boldsymbol{a}$):

Newton's Law: *For two isolated bodies, the ratio of the magnitudes of the acceleration vectors is constant:*

$$\frac{|\boldsymbol{a}_2|}{|\boldsymbol{a}_1|} = k_{21} = \text{const.}, \tag{1}$$

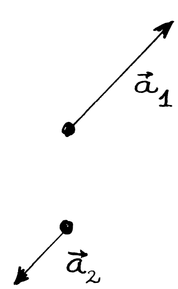
the vectors are parallel

$$\boldsymbol{a}_1 \| \ \boldsymbol{a}_2,$$

and oriented in the opposite directions. In the case of zero accelerations, $\boldsymbol{a}_1 = 0 \Leftrightarrow \boldsymbol{a}_2 = 0$.

An alternative way of writing the second and third conditions in Newton's law is simply:

$$\hat{\boldsymbol{a}}_1 = -\hat{\boldsymbol{a}}_2.$$



We can also write the whole Newton's law as a single equation:

$$\boldsymbol{a}_2 = -k_{21}\boldsymbol{a}_1$$

where k_{21} is a positive constant (we did not have to say that the constant is positive in Eq. (1) since the ratio of magnitudes is always positive). Indeed, this equation implies that the acceleration vectors are parallel and oriented in the opposite directions, whereas the equality of the magnitudes of both sides gives Eq. (1).

The ratio of the magnitudes of accelerations is constant for two given bodies, but takes on a different constant value if one of the bodies is replaced by another body. Let us consider a set of bodies denoted by the subscript $i = 0, 1, 2, \ldots, n$. We can write:

$$\frac{|\boldsymbol{a}_1|}{|\boldsymbol{a}_0|} = k_{10}$$

$$\frac{|\boldsymbol{a}_2|}{|\boldsymbol{a}_0|} = k_{20}$$

$$\cdots$$

$$\frac{|\boldsymbol{a}_n|}{|\boldsymbol{a}_0|} = k_{n0}.$$

Notice that the constants k_{i0} and k_{j0} determine the constant k_{ij}:

$$k_{ij} = \frac{|\boldsymbol{a}_i|}{|\boldsymbol{a}_j|} = \frac{k_{i0}|\boldsymbol{a}_0|}{k_{j0}|\boldsymbol{a}_0|} = \frac{k_{i0}}{k_{j0}}. \tag{2}$$

Thus, taking body 0 as the reference body, the quantity k_{i0} defines a specific and unique property of body i in interaction with any other body j. It is important to understand that although k_{ij} is a constant involving two bodies, k_{i0} is a specific property of body i, The knowledge of k_{i0} and k_{j0} allows one to determine the interaction of body i with body j, as shown above in Eq. (2).

While we could have used k_{i0}'s to describe the dynamics, it turns out that their inverses, m_i, defined as

$$m_i \equiv m_{i0} = 1/k_{i0},$$

and called masses, are more convenient. We have therefore

$$\frac{|\boldsymbol{a}_j|}{|\boldsymbol{a}_i|} = \frac{k_{j0}}{k_{i0}} = \frac{\frac{1}{m_j}}{\frac{1}{m_i}} = \frac{m_i}{m_j} \qquad \text{or} \qquad m_i|\boldsymbol{a}_i| = m_j|\boldsymbol{a}_j|. \tag{3}$$

Assuming the mass of one of the bodies to be the unit mass, we can measure all the other masses relative to this one (for example, the standard of a kilogram). We therefore have:

Definition 1 *If the mass of body 0 is equal to unity, the mass of body 1 is uniquely defined by Newton's Law as*

$$m_1 = \frac{|\boldsymbol{a}_0|}{|\boldsymbol{a}_1|}.$$

Notice that although all the masses are determined relative to body 0, due to the relation (2) one obtains the correct mass ratio for any two bodies, as in Eq. (3). The definition of mass allows us to write Newton's Law in a concise form:

$$\boldsymbol{a}_2 = -k_{21}\boldsymbol{a}_1 \Rightarrow \boldsymbol{a}_2 = -\frac{m_1}{m_2}\boldsymbol{a}_1$$

Newton's Law:

$$m_1\boldsymbol{a}_1 = -m_2\boldsymbol{a}_2. \tag{4}$$

This form in turn makes it possible to introduce the concept of force:

Definition 2 *The force is defined as*

$$\boldsymbol{F} = m\boldsymbol{a}. \tag{5}$$

One should notice that Newton's Law implicitly states (some textbooks state this explicitly) that the ratio of the magnitudes of accelerations does not depend on how strongly the two bodies interact, i.e., what are the forces these bodies act on each other with. In fact, as the bodies move, the distance between them changes all the time and therefore, in general, the force acting on each body changes as well. Still, the ratio $|\boldsymbol{a}_2|/|\boldsymbol{a}_1|$ remains constant at all times. The force can even change with time for a given separation between the bodies (imagine, for example, two bodies connected by a spring which looses its elasticity with time). This independence of the ratio on the forces allows us to determine masses from measurements of accelerations.

With the definition of force, we can also write Eq. (4) as $\boldsymbol{F_1} = -\boldsymbol{F_2}$. Then the ratio of accelerations is

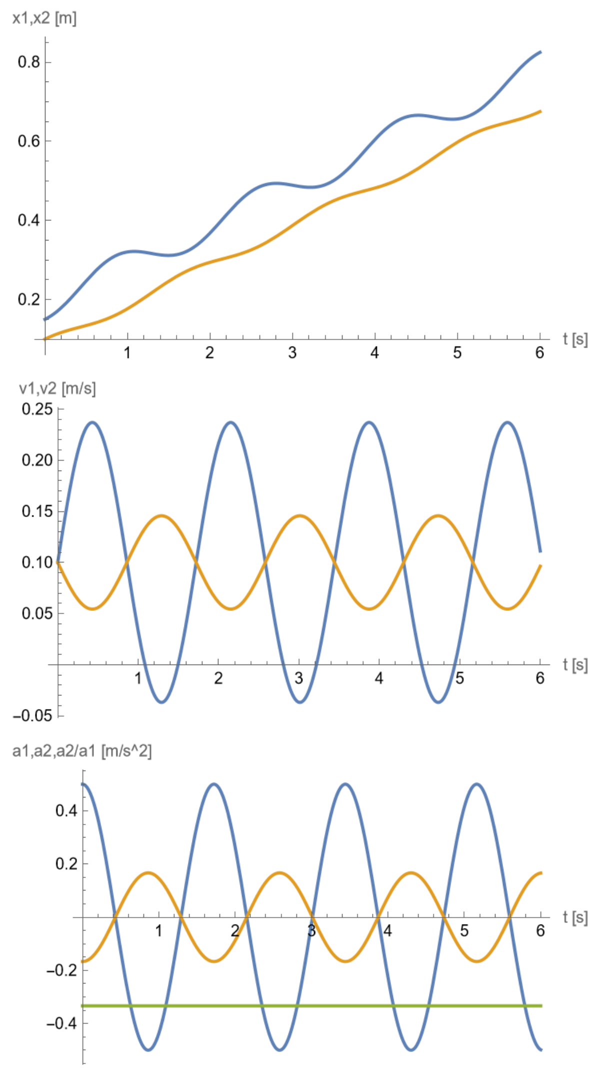
$$\frac{|\boldsymbol{a}_2|}{|\boldsymbol{a}_1|} = \frac{\frac{F_2}{m_2}}{\frac{F_1}{m_1}},$$

where $F_i = |\boldsymbol{F}_i|$, and the forces cancel on the right hand side. This shows that the formulation given above is true also for the gravitational force, which itself depends on the masses of interacting bodies.

We can now show the equivalence between our approach and the traditional form of Newton's Laws. The Newton's Law as formulated above is related to the traditional Newton's Third Law: $\boldsymbol{F_1} = -\boldsymbol{F_2}$. Our definition of the force, Eq. (5), is the traditional Newton's Second Law: $m\boldsymbol{a} = \boldsymbol{F}$. The traditional Newton's First Law is a trivial theorem resulting from the definition of the force, Eq. (5): if $\boldsymbol{a} = 0$ (or $\boldsymbol{v} = \text{const.}$) $\Leftrightarrow \boldsymbol{F} = 0$. In summary:

$$\boldsymbol{a} = 0 \Leftrightarrow \boldsymbol{F} = 0 \qquad m\boldsymbol{a} = \boldsymbol{F} \qquad \boldsymbol{F_1} = -\boldsymbol{F_2} \tag{6}$$

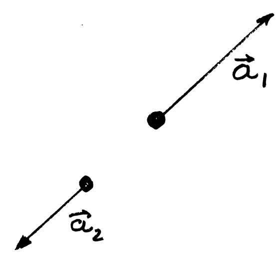
In many cases, the forces acting on a body are known. For example, we can asume that the two carts in the "push-pull" experiment discussed earlier are connected by a harmonic spring, with the spring constant $k = 1$ kg/s^2 and the equilibrium length $l = 0.1$ m, which exerts the force $F = -k(x - l)$, where x is the length of the spring. The masses of the carts are 0.1 and 0.3 kg. With the following initial conditions: $x_1(0) = 0$, $x_2(0) = 0.15$ m, $v_1(0) = 0.1$ m/s, and $v_2(0) = 0.1$ m/s, solutions of Newton's equations result in the following motions: Clearly, the spring in the push-pull experiment discussed earlier is highly



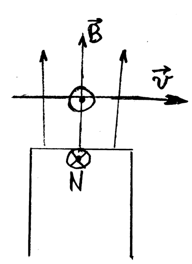
nonlinear.

III. FURTHER DISCUSSION OF NEWTON'S LAW

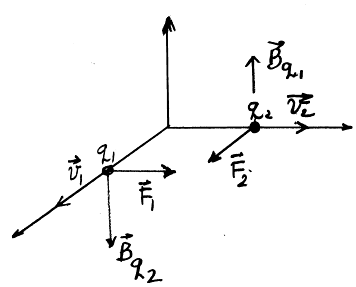
In its most general form, Newton's Law specifies that the accelerations, and therefore the forces, are parallel but not collinear. However, in most cases in classical mechanics the



accelerations and the forces will be collinear, as on the figure above. We refer to these as *central* forces. The more general form of Newton's Law is sometimes called the "weak" form. The form applying to central forces is called the "strong" form. The main example of noncentral forces is the magnetic part of the Lorentz' force $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$, where q is the charge of the particle, $\boldsymbol{E}$ is the electric field, and $\boldsymbol{B}$ is the magnetic field. If a charged particle is moving near to the pole of a magnet, as shown in the figure, the Lorentz force will be acting out of the page. The corresponding force on the pole of the magnet will be—according to Newton's law—into the page. Thus, the two forces are parallel but not colinear.



The magnetic Lorentz force can also violate Newton's Law even if the charges move at low velocities. To see this, consider two charges moving in the absence of any external fields, the setup shown below. An exact calculation of the magnetic fields due to the motion of each charge is complicated, but a simple argument viewing the charges as a part of a linear current gives the correct direction of $\boldsymbol{B}$ ($\boldsymbol{B}_{q2}$ is the field at the position of charge 1 due to charge 2). Then the use of the Lorentz formula gives the forces as shown. Clearly, the forces are not colinear. Why nobody seems to be worried about this violation of Newton's Law? The reason is that at low speeds the electric component of the Lorentz force is so much larger than the magnetic force that the latter is practically unmeasurable. For larger speeds, we are going beyond the range of applicability of classical mechanics.



IV. FRAMES OF REFERENCE, SPACE AND TIME, GALILEAN RELATIVITY

The position vectors $\boldsymbol{r}(t)$ have to be measured in some coordinate system. The choice of this system is not arbitrary, and the definition of appropriate systems, called inertial frames of reference, is "operational":

Definition 3 *An inertial frame of reference is a frame where Newton's law is valid.*

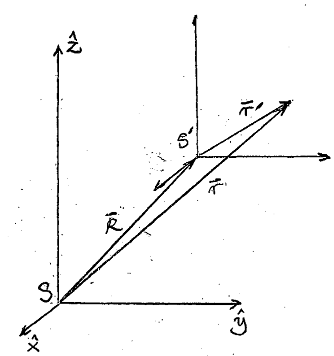
Examples of inertial and noninertial frames:

- Braking train—we know from everyday experience that forces act on us despite staying at rest with respect to the car.
- Turntable—clearly not an inertial frame. If you "sit" on a turntable, at rest with respect to the turntable, a force is needed to maintain this position.
- Earth—in most experiments, Earth is assumed to be an inertial frame. However, Earth is in rotational motion, like a turntable. Thus, it is only approximately an inertial system and departures from inertiality can be measured.

- Star-fixed frame—better than Earth, but still an approximation.

The difficulty of classical mechanics to find an absolute inertial system is resolved in relativistic mechanics which requires no intertial frames. For our purposes, the adoption of Earth as an inertial frame of reference will be in most cases a very good approximation.

Once we know one inertial reference frame, we can find many other ones. Based on experimental observations, we assume that the *space is isotropic*, i.e., if frame S is inertial and frame S' differs from S only by rotation, Newton's Law will hold also in the rotated frame (the angular momentum conservation theorem that we will study later is a consequence of the isotropy of space). Other inertial frames are determined by the Galilean (or Newtonian) relativity (or invariance) principle which tells us that



Theorem 1 *If frame S is inertial and frame S' moves in a uniform motion with respect to S, then S' is an inertial frame as well.*

The proof of this theorem is quite obvious. Denote the constant velocity of S' relative to S by $\boldsymbol{v}_0$ and the vector connecting the centers of systems at $t = 0$ by $\boldsymbol{R}_0$. Thus, at time t, the vector $\boldsymbol{R}$ connecting the centers (see the figure above) is

$$\boldsymbol{R} = \boldsymbol{R}_0 + \boldsymbol{v}_0 t.$$

Assume that the axes of system S' are parallel to the respective axes of system S. Consider a point $\boldsymbol{r}$ which has coordinates $\boldsymbol{r} = [x, y, z]$ in S and $\boldsymbol{r}' = [x', y', z']$ in S' (note that here primes do not indicate derivatives). From the figure, we have

$$\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{r}' = \boldsymbol{R}_0 + \boldsymbol{v}_0 t + \boldsymbol{r}'$$

and therefore the derivatives are

$$\dot{\boldsymbol{r}} = \boldsymbol{v}_0 + \dot{\boldsymbol{r}}'; \qquad \ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}'.$$

Since Newton's Law depends only on second derivatives, this proofs the theorem. A corollary of this theorem is that Newton's Law is the same in all coordinate systems shifted with respect to each other. In the proof given above we have assumed that the axes of systems S and S' are parallel. If not, we first use the isotropy of space to define a system S" with the same origin as S', but with axes parallel to S.

Another principle of this type concerns *homogeneity of time*: experimental results will be the same if an identical experiment is repeated at a later time.

V. CONSERVATION OF MOMENTUM THEOREM

The momentum $\boldsymbol{p}$ of a body is defined as

$$\boldsymbol{p} = m\frac{d\boldsymbol{r}}{dt} = m\boldsymbol{v}.$$

This allows one to write the traditional Newton's Second Law as

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}.$$

For two isolated bodies, i.e., interacting only with each other in absence of any external forces, we have from the traditional Newton's Third Law

$$m_1\boldsymbol{a}_1 = -m_2\boldsymbol{a}_2$$

or

$$m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2 = 0.$$

Therefore,

$$\frac{d\boldsymbol{p}_1}{dt} + \frac{d\boldsymbol{p}_2}{dt} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{p}_1 + \boldsymbol{p}_2\right) = 0.$$

However, this means that

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = \text{const.} \tag{7}$$

This equation defines the conservation of the total momentum $\boldsymbol{P}$ theorem. The latter quantity is defined just as the sum of the individual momenta. As we could see, this theorem is a simple consequence of the traditional Newton's Third Law and it in fact can be viewed as an alternative formulation of this law.

The conservation of momentum theorem extends to an arbitrary number of bodies. Each body obeys Newton's equation

$$\dot{\boldsymbol{p}}_i = \sum_{j=1,\, j\neq i}^{N} \boldsymbol{F}_{ij} + \boldsymbol{F}_i^{\text{ext}}$$

where $\boldsymbol{F}_{ij}$ is the force exerted on body i by body j and $\boldsymbol{F}_i^{\text{ext}}$ is some external force acting on body i. Adding all equations together, we get

$$\frac{d}{dt}\sum_{i=1}^{N}\boldsymbol{p}_i = \sum_{i=1}^{N}\left[\sum_{j=1,\,j\neq i}^{N}\boldsymbol{F}_{ij} + \boldsymbol{F}_i^{\text{ext}}\right].$$

Due to third traditional Newton's law, $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$, so the terms in the first sum cancel in pairs and the sum is zero. Therefore, if the total external force is zero, we have

$$\frac{d}{dt}\sum_{i=1}^{N}\boldsymbol{p}_i = 0 \quad \Rightarrow \quad \boldsymbol{P} = \sum_{i=1}^{N}\boldsymbol{p}_i = \text{const.} \qquad \left(\sum_{i=1}^{N}\boldsymbol{F}_i^{\text{ext}} = 0\right).$$

The argument about cancellations of internal forces can be made more rigorous by writng the sum of these forces as

$$\sum_{i=1}^{N}\sum_{j=1,\,j\neq i}^{N}\boldsymbol{F}_{ij} = \sum_{i=1}^{N}\sum_{j=1,\,j<i}^{N}\boldsymbol{F}_{ij} + \sum_{i=1}^{N}\sum_{j=1,\,j>i}^{N}\boldsymbol{F}_{ij},$$

where we just split each sum over j, which for a given i contains elements $j = 1, 2, \ldots, i - 1, i + 1, \ldots, N$ into the part up to $i - 1$ and the part from $i + 1$ to N. Since variables in summations are dummy, we can exchange i with j in the second sum:

$$\sum_{i=1}^{N}\sum_{j=1,\,j<i}^{N}\boldsymbol{F}_{ij} + \sum_{j=1}^{N}\sum_{i=1,\,i>j}^{N}\boldsymbol{F}_{ji} = \sum_{i=1}^{N}\sum_{j=1,\,j<i}^{N}\boldsymbol{F}_{ij} - \sum_{j=1}^{N}\sum_{i=1,\,i>j}^{N}\boldsymbol{F}_{ij},$$

where we used the third Newton's law. Now we have to realize that both double sums, although they look different, go over the same set of pairs ij. First, see this for $N = 3$. For the first sum we have elements: 21, 31, 32, For the second sum, ji pairs are 12, 13, 23, so ij pairs are 21, 31, 32. Therefore, the sums cancel, proving the theorem. To generalize it to an arbitrary N, think of an $N \times N$ matrix. Both sums add all elements from the triangle below the diagonal: the first sum goes row by row, the second sum goes column by column.

VI. FORCES

So far we looked at forces as just one way to describe relative motions of two bodies. The origins of the forces are an important subject of past and current research in physics. In our course, we will not need to get into any details of theories of forces. Based on observations, we assume that there are only four fundamental forces in nature:

- gravitational
- electromagnetic
- strong
- weak

Recent astronomical observations indicate that there may exist one more force, sometimes called the dark force (related to dark energy), which is responsible for the accelerated expansion of the universe. In our course we will encounter only the first two forces on the list above. The strong force acts between nucleons (protons and neutrons). The weak force is responsible for the β decay of nuclei.

VII. GRAVITATIONAL FORCE

The gravitational force acts between any two bodies and for pointwise bodies is given by the Newton's law of gravity:

$$|\boldsymbol{F}| = G\frac{m_1 m_2}{r^2}$$

where G is a constant and r is the distance between the bodies. The gravitational force is central and always attractive. This is the force we experience everyday. Note that the gravitational attraction between a point near Earth's surface and the Earth is not between pointwise bodies, but one may show that this force is the same as if the whole mass of the Earth was located in its center (one can prove, without any approximations, that the interaction of a point mass with a uniform density sphere of mass M is the same as the interaction with a point mass M placed in the center of the sphere). Thus, we may write for a body of mass m interacting with the Earth

$$|\boldsymbol{F}| = G\frac{Mm}{r^2} = mg$$

where M is the mass of the Earth. For a body near the surface of the Earth, the distance r is nearly constant so that the quantity $g = GM/r^2$ is nearly constant and it is called gravitational acceleration.

VIII. LORENTZ FORCE

The force due to electric, $\boldsymbol{E}$, and magnetic, $\boldsymbol{B}$, fields acting on a body of charge q moving with velocity $\boldsymbol{v}$ is given be the so-called Lorentz' force

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right).$$

IX. DERIVATIVE FORCES

Several other forces which will appear in our course are not new fundamental forces but are macroscopic effects of electromagnetic forces acting on microscopic distances. Among them are:

- normal (reaction)
- frictional
- retarding (drag)

forces.

The normal force is responsible for an object placed on the surface of the table to be at rest despite the action of the gravitational force. The origin of this force are the interactions between atoms forming the table. The frictional force is due to interatomic interactions between two surfaces touching each other. Such interactions are overall attractive, therefore force is needed to move one surface with respect to the other. The retarding forces of fluids (gases or liquids) are due to the same mechanism. The frictional and drag forces are dissipative, i.e., always decrease mechanical energy.