Classical Mechanics Lecture Notes: Tensor of Inertia Summary

Here is a summary of recent lectures. All theorems were rigorously proved (unless stated otherwise).

Definition of tensor of inertia

Consider a single body of mass m rotating around an axis $\boldsymbol{\omega} = \sum \omega_i \hat{\boldsymbol{x}}_i$ through the center of coordinate system. The angular momentum of such motion is given by

$$[L] = \mathbb{I}[\omega],$$

where $\boldsymbol{L} = \sum \text{L}_i \hat{\boldsymbol{x}}_i$, $[L] = [L_1 \; L_2 \; L_3]^T$, $[\omega] = [\omega_1 \; \omega_2 \; \omega_3]^T$, and $\mathbb{I}$ is a 3x3 matrix with elements $I_{ii} = m(r^2 - x_i^2)$ and $I_{ij} = -m x_i x_j$. For a rigid body, the tensor of inertia is defined by dividing the body into small pieces with mass Δm_i, summing the angular momenta, and taking the limit of $\Delta m_i \to 0$.

Eigenvalues and eigenvectors of $\mathbb{I}$

If there exists a vector $\boldsymbol{\omega}$ such that

$$\mathbb{I}[\omega] = \lambda[\omega],$$

where λ is a number, λ is called an eigenvalue of $\mathbb{I}$ and $[\omega]$ is called an eigenvector of $\mathbb{I}$. For a $N \times N$ symmetric matrix, there exists N real eigenvalues (some of them can be identical) and N linearly independent eigenvectors (this theorem was not proved). In the case of the inertia tensor, the eigenvalues are called principal moments of inertia and the eigenvectors are called principal axes of inertia. Thus, there are 3 such moment and axes

$$\mathbb{I}[\omega]_i = \lambda_i[\omega]_i, \quad i = 1, 2, 3,$$

with notation $[\omega]_i = [\omega_{i1} \; \omega_{i2} \; \omega_{i3}]^T$ or $\boldsymbol{\omega}_i = \sum \omega_{ij} \hat{\boldsymbol{x}}_j$.

Rotation about a principal axis

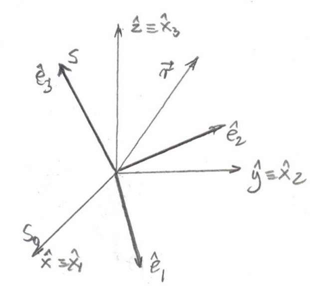
$$[L] = \mathbb{I}[\omega]_i = \lambda_i[\omega]_i.$$

Since λ_i is a fixed real number, $\boldsymbol{L} \parallel \boldsymbol{\omega}_i$. Thus, $\boldsymbol{L}$ is constant, therefore $\mathbb{I}$ is constant as well.

Orthonormality of principal axes of inertia

We have shown that eigenvectors corresponding to different eigenvalues are orthogonal and those corresponding to a degenerate eigenvalue can be orthogonalized. Thus.

$$\boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_j = \delta_{ij}.$$



We will assume from now on that the eigenvectors are normalized and write $\hat{\boldsymbol{\omega}}_i = \sum \omega_{ij} \hat{\boldsymbol{x}}_j$. We will also often use $\hat{\boldsymbol{\omega}}_i = \hat{\boldsymbol{e}}_i$ to avoid confusion when later we will discuss also arbitrary rotation axes $\boldsymbol{\omega}$. The inertial space-fixed frame S_0, the body-fixed frame S, and the principal axes of rotation defining S are marked on the figure. The term "body-fixed" comes from the principal axes of rotation being fixed relative to the body.

Rotation around arbitrary $\boldsymbol{\omega}$ via rotations around principal axes

The vector of an arbitrary rotation $\boldsymbol{\omega} = \sum \omega_i \hat{\boldsymbol{x}}_i$ can be expressed in terms of eigenvectors $\hat{\boldsymbol{e}}_i$: $\boldsymbol{\omega} = \sum \omega'_i \hat{\boldsymbol{e}}_i$. Therefore

$$[L] = \mathbb{I}[\omega] = \mathbb{I} \sum \omega'_i [e]_i = \sum \lambda_i \omega'_i [e]_i .$$

Notice that all coordinates in $[\dots]$ are in S_0.

Transition matrix $\mathbb{S}$

The transition matrix is defined as

$$\mathbb{S} = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix}$$

and it has the properties

$$\mathbb{S}\mathbb{S}^T = \mathbb{S}^T \mathbb{S} = \mathbb{1}$$

$$\mathbb{S}\mathbb{I}\mathbb{S}^T = \mathbb{I}_{\text{diag}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Coordinates of positions vector in S_0 and S

Consider a vector $\boldsymbol{r}$ in S_0 and in S (defined by principal axes)

$$\boldsymbol{r} = \sum_i x_i \hat{\boldsymbol{x}}_i = \sum_i x'_i \hat{\boldsymbol{e}}_i .$$

Defining $[r] = [x_1 \; x_2 \; x_3]^T$ and $[r'] = [x'_1 \; x'_2 \; x'_3]^T$, we have

$$[r'] = \mathbb{S}[r].$$

Angular momentum in frame S

The derivation of the transformation between $[r]$ and $[r']$ is actually valid for any vector, in particular for the angular momentum vector

$$\boldsymbol{L} = \sum_i L_i \hat{\boldsymbol{x}}_i = \sum_i L'_i \hat{\boldsymbol{e}}_i ,$$

and also for the arbitrary rotation vector $\boldsymbol{\omega}$. Therefore we have

$$[L'] = \mathbb{S}[L] = \mathbb{S}\mathbb{I}[\omega] = \mathbb{S}\mathbb{I}\mathbb{S}^T\mathbb{S}[\omega] = \mathbb{I}_{\text{diag}}[\omega'].$$

Thus

$$L'_i = \lambda_i \omega'_i.$$

Tensor $\mathbb{I}$ in frame S

The inertia tensor for a pointlike body of mass m can be written as

$$\mathbb{I} = m\left([r]^T[r]\mathbb{1} - [r][r]^T\right)$$

From this follows that the inertia tensor computed in frame S:

$$\mathbb{I}' = m\left([r']^T[r']\mathbb{1} - [r'][r']^T\right) = \mathbb{I}_{\text{diag}}.$$