# Classical Mechanics Lecture Notes
# ANGULAR MOMENTUM

September 9, 2024

## I. CENTER OF MASS

We will start from defining a center of mass (CM). When considering rotational motion of several bodies, it is often convenient to express positions of the bodies wrt its CM.

Consider a set of point masses $m_i$, $i = 1, \ldots, N$. The center of mass (CM) of such system is defined as

$$\boldsymbol{R}_{\mathrm{CM}} = \frac{1}{M} \sum_{i=1}^{N} \boldsymbol{r}_i m_i,$$

where $M$ is the sum of all masses. To understand this concept, first consider two bodies of equal mass. Clearly, $\boldsymbol{R}_{\mathrm{CM}}$ is in the midpoint between these bodies. Now consider the case when one of the masses is much larger than the other one. $\boldsymbol{R}_{\mathrm{CM}}$ is now very close to the large mass and still on the line connecting the two objects. For example, the CM of the Earth-Moon system is 4,600 km from the center of the Earth.

The center of mass is related to the total angular momentum $\boldsymbol{P}$ defined earlier

$$M\dot{\boldsymbol{R}} = \sum_{i=1}^{N} \dot{\boldsymbol{r}}_i m_i = \sum_{i=1}^{N} \boldsymbol{p}_i = \boldsymbol{P}$$

(we will omit the subscript 'CM' from now on in most cases: $\boldsymbol{R}_{\mathrm{CM}} \equiv \boldsymbol{R}$). Taking the second derivative and using Newton's equation for each body, we get

$$M\ddot{\boldsymbol{R}} = \dot{\boldsymbol{P}} = \sum_{i=1}^{N} \ddot{\boldsymbol{r}}_i m_i = \sum_{i=1}^{N} \left[ \sum_{j=1,\, j\neq i}^{N} \boldsymbol{F}_{ij} + \boldsymbol{F}_i^{\mathrm{ext}} \right] = \boldsymbol{F}^{\mathrm{ext}}$$

where we used the fact proven recently that the sum of the internal forces is zero. Thus,

$$\dot{\boldsymbol{P}} = M\ddot{\boldsymbol{R}} = \boldsymbol{F}^{\mathrm{ext}},$$

i.e., the motion of the center of mass of a system of particles is determined by the sum of the external forces. In particular, if this sum is zero, CM is at rest or moves with a uniform velocity.

The definition formulated above can be extended to a rigid body. Consider first a model of rigid body consisting of point masses $m_i$, $i = 1, \ldots, N$ connected by rigid massless rods.

In this case, the definition given above holds. For a continuous rigid body, this definition becomes

$$\boldsymbol{R}_{\mathrm{CM}} = \frac{1}{M}\int_{\mathrm{body}} \boldsymbol{r}\, dm = \frac{1}{M}\int_{\mathrm{body's\ volume}} \boldsymbol{r}\rho\, dV$$

where $\rho$ is the density at point $\boldsymbol{r}$. Note that $\boldsymbol{r}$ can be measured in any coordinate system and the CM vector is then expressed in the same coordinate system.

## II. ANGULAR MOMENTUM

For a point mass particle, the angular momentum, $\boldsymbol{l}$, is defined as:

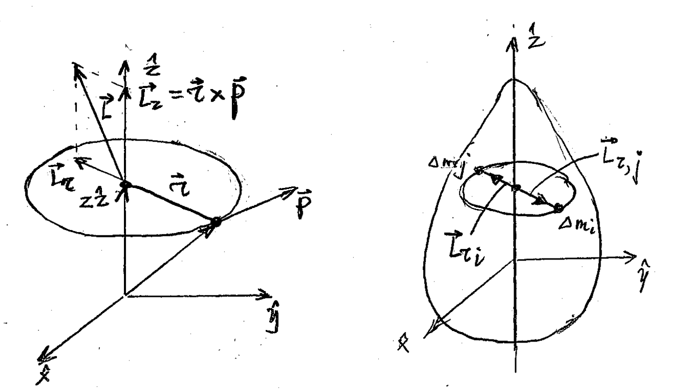
$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p}.$$

This definition holds for any type of motion, including motion along straight line. For a set of particles (not necessarily connected in any way) the total $\boldsymbol{L}$ is defined as

$$\boldsymbol{L} = \sum_{i=1}^{N} \boldsymbol{l}_i.$$

Again, the particles are not necessarily rotating.

Even if the particle is in a circular motions around an axis fixed in space along $\hat{\boldsymbol{z}}$ axis, the angular momentum is not directed along this axis, see the figure. It will only be along $\hat{\boldsymbol{z}}$ if the circle is in the $xy$ plane. Another such case is the rotation of a body along its symmetry axis, see the picture below on the right. Here the $xy$ components of symmetrically located $\boldsymbol{l}_i$ and $\boldsymbol{l}_j$ cancel at all times.



### A. Newton's equation for angular momentum for a set of particles

For a single particle, starting from the time derivative of $\boldsymbol{l}$ and using Newton's equation, we get

$$\frac{d\boldsymbol{l}}{dt} = m\left[\frac{d\boldsymbol{r}}{dt} \times \boldsymbol{v} + \boldsymbol{r} \times \frac{d\boldsymbol{v}}{dt}\right] = m\boldsymbol{r} \times \boldsymbol{a} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{\Gamma},$$

since $\boldsymbol{v} \times \boldsymbol{v} = 0$. The cross-product of $\boldsymbol{r}$ and $\boldsymbol{F}$ is called torque. The final equation,

$$\frac{d\boldsymbol{l}}{dt} = \boldsymbol{\Gamma},$$

is called Newton's equation (or law) for angular momentum.

In the derivation above, we calculated the derivative of $\boldsymbol{r} \times \boldsymbol{p}$ as it were a product of two functions. That this is correct can be seen by writing the cross product using the definition via determinant

$$\frac{d}{dt}\left[(yp_z - zp_y)\hat{\boldsymbol{x}} + (zp_x - xp_z)\hat{\boldsymbol{y}} + (xp_y - yp_z)\hat{\boldsymbol{z}}.\right]$$

Thus, indeed we have a sum of products.

In special cases, the angular momentum vector can be constant in time, as it will be discussed next. In a general case, the vector $\boldsymbol{l}$ is not pointed along any fixed direction, but changes the direction in time, as evident from the equation above.

This expression is immediately extended to $N$ particles

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{\text{int}} + \boldsymbol{\Gamma}^{\text{ext}}, \tag{1}$$

where the total torque is $\boldsymbol{\Gamma} = \sum_{i=1}^{N} \boldsymbol{\Gamma}_i$. It can be separated into an internal

$$\boldsymbol{\Gamma}^{\text{int}} = \sum_{i=1}^{N} \boldsymbol{r}_i \times \sum_{j=1,\, j\neq i}^{N} \boldsymbol{F}_{ij}$$

and external

$$\boldsymbol{\Gamma}^{\text{ext}} = \sum_{i=1}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_i^{\text{ext}}$$

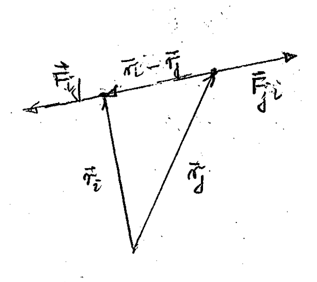
contributions.

### B. Angular momentum conservation

For a single particle, if $\boldsymbol{\Gamma} = 0$ then

$$\frac{d\boldsymbol{l}}{dt} = 0 \quad \Rightarrow \quad \boldsymbol{l} = \text{const.}$$

The torque is of course zero if the net force acting on a body is zero. The other possibility is that the net force $\boldsymbol{F}$ is parallel to $\boldsymbol{r}$.



Similarly, if the total torque $\boldsymbol{\Gamma} = 0$ then

$$\boldsymbol{L}(t) = \text{const.}$$

Here in addition to the cases for a torque be zero for each single particle, the torque is zero if the sum of all torques due to the external forces is zero. The reason is that the torques from internal forces cancel out, similarly as in the case of the linear momentum conservation the internal forces cancel out. Despite this similarity, the cancellation for torques does not follow exclusively from the third Newton's law and in fact it true only for collinear forces. The proof is as follows. From the equation for $\boldsymbol{\Gamma}^{\text{int}}$

$$\boldsymbol{\Gamma}^{\text{int}} = \sum_i \sum_{j \neq i} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} = \sum_i \sum_{j>i} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} + \sum_i \sum_{j<i} \boldsymbol{r}_i \times \boldsymbol{F}_{ij}$$

To understand how the sum was split, construct a square matrix of forces. Then the first (second) sum includes elements from the upper (lower) triangle. Now notice that

$$\sum_i \sum_{j<i} a_{ij} = \sum_i \sum_{j>i} a_{ji}$$

and therefore we can write

$$\boldsymbol{\Gamma}^{\text{int}} = \sum_i \sum_{j>i} \left(\boldsymbol{r}_i \times \boldsymbol{F}_{ij} + \boldsymbol{r}_j \times \boldsymbol{F}_{ji}\right).$$

Now use Newton's third law in the second term and then pull out $\boldsymbol{F}_{ij}$

$$\boldsymbol{\Gamma}^{\text{int}} = \sum_i \sum_{j>i} \left(\boldsymbol{r}_i \times \boldsymbol{F}_{ij} - \boldsymbol{r}_j \times \boldsymbol{F}_{ij}\right) = \sum_i \sum_{j>i} \left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) \times \boldsymbol{F}_{ij} = 0.$$

If the internal forces are collinear, the vector $\boldsymbol{r}_i - \boldsymbol{r}_j$ is parallel to $\boldsymbol{F}_{ij}$, as shown on the figure, and the cross product is zero. Thus, we have shown that the sum of all torques due to internal collinear forces is zero.

An important case is when there are no external forces acting on a system of particles, i.e., the system is isolated. For isolated systems, both the linear and angular momentum are always conserved.