Class 20: Dirac Notation

All quantum states are described by vectors in some linear space. These vectors are called state vectors. In Dirac notation, these vectors are described by a ket. Examples of kets are $|n\rangle$ and $|p\rangle$ where $n$ is a quantum number associated with a stationary state e.g. of the infinite square well or harmonic oscillator and $p$ may indicate a momentum eigenvalue. For simultaneous eigenstates of two commuting operators, the ket could be $|n, l\rangle$. A ket $|\psi\rangle$ is used if the state vector is a superposition of eigenstates.

Associated with the vector space of kets is a dual space of linear functionals called bras and denoted by $\langle a |$ where $a$ is an appropriate label. Each ket has a corresponding bra. [In this context, the application of a linear functional on a function gives a number. The functional $F$ corresponding to a function $f$ is such that $F(g) = \langle f | g \rangle$, for all functions $g$.]

The inner (scalar) product of a bra and a ket is denoted by the bracket (bra – ket) symbol $\langle a | b \rangle$. From the properties of the inner product, we have

$$\langle \psi | \phi \rangle^* = \langle \phi | \psi \rangle,$$

(20.1)

and

$$\langle \psi | \alpha \phi_1 + \beta \phi_2 \rangle = \alpha \langle \psi | \phi_1 \rangle + \beta \langle \psi | \phi_2 \rangle.$$

(20.2)

When an operator acts on a state vector, it gives another state vector. This is written in two equivalent ways:

$$\hat{A} | \psi \rangle = | \hat{A} \psi \rangle.$$

(20.3)

It follows that

$$\langle \phi | \hat{A} | \psi \rangle = \langle \phi | \hat{A} \psi \rangle.$$

(20.4)

The Hermitian conjugate $\hat{A}^*$ of the operator $\hat{A}$ is defined by

$$\langle \phi | \hat{A}^* | \psi \rangle = \langle \psi | \hat{A} | \phi \rangle^*.$$

(20.5)

A general state vector can be written as a linear combination of orthonormal basis state vectors (these could be the eigenfunctions of the Hamiltonian operator):

$$| \psi \rangle = \sum_{n=1}^{\infty} c_n | n \rangle.$$  

(20.6)
The orthonormality condition is

\[ \langle m | n \rangle = \delta_{mn}. \]  \hspace{1cm} (20.7)

Hence

\[ \langle m | \psi \rangle = \sum_{n=1}^{\infty} c_n \langle m | n \rangle = c_m. \]  \hspace{1cm} (20.8)

Using this in equation (20.6), we see that

\[ |\psi\rangle = \sum_{n=1}^{\infty} |n\rangle \langle n | \psi \rangle. \]  \hspace{1cm} (20.9)

Since \( |\psi\rangle \) is quite general, we must have

\[ \sum_{n=1}^{\infty} |n\rangle \langle n | = 1, \]  \hspace{1cm} (20.10)

where \( 1 \) is the identity operator. Equation (20.10) is the completeness relation (for basis vectors with a discrete spectrum).

Now consider an operator that has a continuous spectrum, e.g. the momentum operator. The general state vector can be written as

\[ |\psi\rangle = \int_{-\infty}^{\infty} c(p) |p\rangle dp. \]  \hspace{1cm} (20.11)

Now

\[ \langle p' | \psi \rangle = \int_{-\infty}^{\infty} c(p) \langle p' | p \rangle dp = \int_{-\infty}^{\infty} c(p) \delta(p' - p) dp = c(p'). \]  \hspace{1cm} (20.12)

Changing the notation, we see that the momentum space wave function is

\[ \phi(p) = \langle p | \psi \rangle. \]  \hspace{1cm} (20.13)

Similarly the position space wave function is

\[ \psi(x) = \langle x | \psi \rangle. \]  \hspace{1cm} (20.14)

If the operator has a continuous spectrum, the completeness theorem will be of form
\[ \int_{-\infty}^{\infty} \langle p | dp = 1, \quad (20.15) \]

where, for example, \( | p \rangle \) is the ket corresponding to a momentum operator eigenfunction.

Since
\[ \psi(x) = \langle x | \psi \rangle = \langle x | 1 \psi \rangle = \int_{-\infty}^{\infty} \langle x | p \rangle \langle p | \psi \rangle dp = \int_{-\infty}^{\infty} \langle x | p \rangle \phi(p) dp, \quad (20.16) \]
we see that
\[ \langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}. \quad (20.17) \]

This is the momentum operator eigenfunction found earlier.

**Projection operators**

The projection operator \( P_n \) is defined by
\[ P_n = | n \rangle \langle n |. \quad (20.18) \]
 Clearly
\[ \sum_{n=1}^{\infty} P_n = 1. \quad (20.19) \]

Now
\[ P_n | \psi \rangle = | n \rangle \langle n | \psi \rangle. \quad (20.20) \]

Hence the action of \( P_n \) is to project an arbitrary state \( | \psi \rangle \) into state \( | n \rangle \) with probability amplitude \( \langle n | \psi \rangle \). But this is precisely what measurement does! This allows us to write the Hamiltonian as
\[ H = \sum_{n=1}^{\infty} E_n P_n, \quad (20.21) \]
where \( E_n \) is an energy eigenvalue.

**The Schrödinger equation and the Heisenberg picture**

In Dirac notation the Schrödinger equation is
\[ i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle. \]  

(20.22)

This has formal solution

\[ |\psi(t)\rangle = \exp \left( -\frac{i\hat{H}t}{\hbar} \right) |\psi(0)\rangle = \exp \left( -\frac{i\hat{H}t}{\hbar} \right) |\psi(0)\rangle, \]

(20.23)

where the operator is

\[ \exp \left( -\frac{i\hat{H}t}{\hbar} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{i\hat{H}t}{\hbar} \right)^n. \]

(20.24)

Consider the expectation value of an operator \( \hat{Q} \) that does not explicitly depend on time. We have

\[ \langle Q \rangle = \langle \psi(t) | \hat{Q} | \psi(t) \rangle = \langle \psi(t) | \hat{Q} \psi(t) \rangle = \exp \left( -\frac{i\hat{H}t}{\hbar} \right) \hat{Q} \exp \left( -\frac{i\hat{H}t}{\hbar} \right) |\psi(0)\rangle \]

\[ = \langle \psi(0) | \exp \left( \frac{i\hat{H}t}{\hbar} \right) \hat{Q} \exp \left( -\frac{i\hat{H}t}{\hbar} \right) |\psi(0)\rangle = \langle \psi(0) | \exp \left( \frac{i\hat{H}t}{\hbar} \right) \hat{Q} \exp \left( -\frac{i\hat{H}t}{\hbar} \right) |\psi(0)\rangle. \]

(20.25)

Define a new operator \( \hat{Q}_t \) by

\[ \hat{Q}_t = \exp \left( \frac{i\hat{H}t}{\hbar} \right) \hat{Q} \exp \left( -\frac{i\hat{H}t}{\hbar} \right). \]

(20.26)

Now we see that

\[ \langle Q \rangle = \langle \psi(t) | \hat{Q}_t | \psi(t) \rangle = \langle \psi(0) | \hat{Q}_t | \psi(0) \rangle. \]

(20.27)

What this tells us is that there are two equivalent ways of solving problems in quantum mechanics. In the Schrödinger picture the operator does not change with time but the state vector does, whereas in the alternative Heisenberg picture the operator changes with time but the state vector remains the same.

**Example of use of the abstract mathematical nature of Dirac notation**

Example 3.8. Consider a quantum system for which the state vector lies in a vector space of dimension 2. Any state vector \( |\psi\rangle \) can be written as a linear combination of two basis vectors which will be denoted by \( |1\rangle \) and \( |2\rangle \). (The labels are not eigenvalues of the Hermitian operator). Representing the basis vectors by orthonormal column vectors.
\[ |1\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]

a general state vector

\[ |\psi\rangle = a|1\rangle + b|2\rangle, \]

is represented by

\[ |\psi\rangle \rightarrow \begin{pmatrix} a \\ b \end{pmatrix}. \]

In this basis the Hamiltonian operator will a 2×2 Hermitian matrix. Suppose it has specific form

\[ H = \begin{pmatrix} h & g \\ g & h \end{pmatrix}, \]

where \( g \) and \( h \) are real constants. The eigenvalue problem for the Hamiltonian operator is

\[ \begin{pmatrix} h & g \\ g & h \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = E \begin{pmatrix} a \\ b \end{pmatrix}, \]

where \( E \) is an eigenvalue. The characteristic equation is

\[ (h - E)^2 - g^2 = 0, \]

which has solutions \( E_\pm = h \pm g \). The eigenvectors are given by

\[ ha + gb = E_\pm a = (h \pm g) a, \]

which gives \( b = \pm a \). Two orthonormal eigenvectors are

\[ |\pm\rangle \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \]

In terms of the basis vectors,

\[ |\pm\rangle = \frac{1}{\sqrt{2}} (|1\rangle \pm |2\rangle). \]

Since
\[
\begin{bmatrix}
1 \\
0
\end{bmatrix} = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right],
\begin{bmatrix}
0 \\
1
\end{bmatrix} = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right],
\]
we see that
\[
|1\rangle = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle),
|2\rangle = \frac{1}{\sqrt{2}} (|+\rangle - |-\rangle).
\]

We can now find the time evolution of any initial state by writing it as a linear combination of the energy eigenvectors and inserting the appropriate time dependencies. E.g. if the initial state is $|1\rangle$, this will evolve according to
\[
|\psi\rangle = \frac{1}{\sqrt{2}} \left( |+\rangle e^{-iE_1 \tau/\hbar} + |-\rangle e^{-iE_2 \tau/\hbar} \right) = \frac{1}{2} \left( (|1\rangle + |2\rangle) e^{-iE_1 \tau/\hbar} + (|1\rangle - |2\rangle) e^{-iE_2 \tau/\hbar} \right)
\]
\[
= \frac{1}{2} \left( (|1\rangle + |2\rangle) e^{-i(h+g)\tau/\hbar} + (|1\rangle - |2\rangle) e^{-i(h-g)\tau/\hbar} \right)
\[
= \frac{1}{2} e^{-i\hbar\tau/\hbar} \left( (|1\rangle + |2\rangle) e^{-i\tau/\hbar} + (|1\rangle - |2\rangle) e^{i\tau/\hbar} \right)
\]
\[
= e^{-i\hbar\tau/\hbar} \left( |1\rangle \cos \frac{gt}{\hbar} - |2\rangle i \sin \frac{gt}{\hbar} \right).
\]

We see that although the system starts out in state $|1\rangle$ after a time $\pi \hbar/2g$ it is purely in state $|2\rangle$. The system oscillates between the two states. This is a simple model for oscillations between two ‘flavors’ of neutrinos. States $|1\rangle$ and $|2\rangle$ would then be eigenstates of the flavor operator.