13 GENERAL THEORY OF LINEAR RESPONSE TO AN EXTERNAL PERTURBATION

Consider an interacting many-particle system with a time-independent hamiltonian $\hat{H}$. The exact state vector in the Schrödinger picture $|\Psi_S(t)\rangle$ satisfies the Schrödinger equation

$$i\hbar\frac{\partial|\Psi_S(t)\rangle}{\partial t} = \hat{H}|\Psi_S(t)\rangle \tag{13.1}$$

with the explicit solution

$$|\Psi_S(t)\rangle = e^{-i\hat{H}t/\hbar}|\Psi_S(0)\rangle \tag{13.2}$$

Suppose that the system is perturbed at $t = t_0$ by turning on an additional time-dependent hamiltonian $\hat{H}^{ex}(t)$. The new Schrödinger state vector $|\bar{\Psi}_S(t)\rangle$ satisfies the modified equation ($t > t_0$)

$$i\hbar\frac{\partial|\bar{\Psi}_S(t)\rangle}{\partial t} = [\hat{H} + \hat{H}^{ex}(t)]|\bar{\Psi}_S(t)\rangle \tag{13.3}$$

and we shall seek a solution in the form

$$|\bar{\Psi}_S(t)\rangle = e^{-i\hat{H}t/\hbar}\hat{A}(t)|\Psi_S(0)\rangle \tag{13.4}$$

where the operator $\hat{A}(t)$ obeys the causal boundary condition

$$\hat{A}(t) = 1 \qquad t \leqslant t_0 \tag{13.5}$$

A combination of Eqs. (13.3) and (13.4) yields the operator equation for $\hat{A}(t)$:

$$\begin{aligned} i\hbar\frac{\partial\hat{A}(t)}{\partial t} &= e^{i\hat{H}t/\hbar}\hat{H}^{ex}(t)\,e^{-i\hat{H}t/\hbar}\hat{A}(t) \\ &\equiv \hat{H}_H^{ex}(t)\,\hat{A}(t) \end{aligned} \tag{13.6}$$

where $\hat{H}_H^{ex}(t)$ is in the usual Heisenberg picture that makes use of the full interacting $\hat{H}$.

Equation (13.6) may be solved iteratively for $t > t_0$

$$\hat{A}(t) = 1 - i\hbar^{-1}\int_{t_0}^{t} dt'\,\hat{H}_H^{ex}(t') + \cdots \tag{13.7}$$

where the causal boundary condition [Eq. (13.5)] is automatically satisfied because $\hat{H}^{ex}(t) = 0$ if $t < t_0$. The corresponding state vector is given by

$$|\bar{\Psi}_S(t)\rangle = e^{-i\hat{H}t/\hbar}|\Psi_S(0)\rangle - i\hbar^{-1}e^{-i\hat{H}t/\hbar}\int_{t_0}^{t} dt'\,\hat{H}_H^{ex}(t')|\Psi_S(0)\rangle + \cdots \tag{13.8}$$

All physical information of interest is contained in matrix elements of Schrödinger picture operators $\hat{O}_S(t)$ (which may depend explicitly on time)

$$\begin{aligned} \langle\hat{O}(t)\rangle_{ex} &\equiv \langle\bar{\Psi}'_S(t)|\hat{O}_S(t)|\bar{\Psi}_S(t)\rangle \\ &= \langle\Psi'_S(0)|\left[1 + i\hbar^{-1}\int_{t_0}^{t} dt'\,\hat{H}_H^{ex}(t') + \cdots\right]e^{i\hat{H}t/\hbar}\hat{O}_S(t)\,e^{-i\hat{H}t/\hbar} \\ &\qquad \times\left[1 - i\hbar^{-1}\int_{t_0}^{t} dt'\,\hat{H}_H^{ex}(t') + \cdots\right]|\Psi_S(0)\rangle \\ &= \langle\Psi'_H(0)|\hat{O}_H(t)|\Psi_H(0)\rangle + i\hbar^{-1}\langle\Psi'_H(0)|\int_{t_0}^{t} dt' \\ &\qquad \times[\hat{H}_H^{ex}(t'),\hat{O}_H(t)]|\Psi_H(0)\rangle + \cdots \end{aligned} \tag{13.9}$$

Only the linear terms in $\hat{H}^{ex}$ have been retained, and the subscript H denotes the Heisenberg picture with respect to the time-independent hamiltonian $\hat{H}$ [compare Eqs. (6.28) and (6.32)]. The first-order change in a matrix element arising from an external perturbation is here expressed in terms of the exact Heisenberg operators of the interacting but unperturbed system. In particular, if $|\Psi_H\rangle$ and $|\Psi'_H\rangle$ both denote the normalized ground state $|\Psi_0\rangle$, the linear response of the ground-state expectation value of an operator is given by

$$\begin{aligned} \delta\langle\hat{O}(t)\rangle &\equiv \langle\hat{O}(t)\rangle_{ex} - \langle\hat{O}(t)\rangle \\ &= i\hbar^{-1}\int_{t_0}^{t} dt'\,\langle\Psi_0|[\hat{H}_H^{ex}(t'),\hat{O}_H(t)]|\Psi_0\rangle \end{aligned} \tag{13.10}$$

As a specific example, consider a system with charge e per particle in the presence of an external scalar potential $\varphi^{ex}(\mathbf{x}t)$, which is turned on at $t = t_0$. The corresponding external perturbation is equal to

$$\hat{H}_H^{ex}(t) = \int d^3x\,\hat{n}_H(\mathbf{x}t)\,e\varphi^{ex}(\mathbf{x}t) \tag{13.11}$$

where $\hat{n}_H$ is the exact particle density operator in the unperturbed system. The linear response may be characterized by the change in the density

$$\begin{aligned} \delta\langle\hat{n}(\mathbf{x}t)\rangle &= i\hbar^{-1}\int_{t_0}^{t} dt'\int d^3x'\,e\varphi^{ex}(\mathbf{x}'t')\langle\Psi_0|[\hat{n}_H(\mathbf{x}'t'),\hat{n}_H(\mathbf{x}t)]|\Psi_0\rangle \\ &= i\hbar^{-1}\int_{t_0}^{t} dt'\int d^3x'\,e\varphi^{ex}(\mathbf{x}'t')\langle\Psi_0|[\tilde{n}_H(\mathbf{x}'t'),\tilde{n}_H(\mathbf{x}t)]|\Psi_0\rangle \end{aligned} \tag{13.12}$$

where we have now introduced the deviation operators $\tilde{n}_H(\mathbf{x}t) \equiv \hat{n}_H(\mathbf{x}t) - \langle\hat{n}_H(\mathbf{x}t)\rangle$ [compare Eq. (12.3)]. (Note that the c numbers always commute.) If the retarded density correlation function is defined in analogy with Eq. (12.5)

$$iD^R(x,x') = \theta(t-t')\frac{\langle\Psi_0|[\tilde{n}_H(x),\tilde{n}_H(x')]|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \tag{13.13}$$

then Eq. (13.12) may be rewritten as

$$\delta\langle\hat{n}(\mathbf{x}t)\rangle = \hbar^{-1}\int_{-\infty}^{\infty} dt'\int d^3x'\,D^R(\mathbf{x}t,\mathbf{x}'t')\,e\varphi^{ex}(\mathbf{x}'t') \tag{13.14}$$

where the causal behavior is enforced by the retarded nature of D^R, and we have used the fact that φ^{ex} vanishes for $t' < t_0$. Equation (13.14) typifies a general result that the linear response of an operator to an external perturbation is expressible as the space-time integral of a suitable retarded correlation function.

If the system is spatially homogeneous, then $D^R(x,x') = D^R(x - x')$, and it is useful to introduce Fourier transforms

$$\varphi^{ex}(\mathbf{k},\omega) \equiv \int d^3x \int dt\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\omega t} \varphi^{ex}(\mathbf{x}t) \tag{13.15}$$

$$\delta\langle\hat{n}(\mathbf{k},\omega)\rangle \equiv \int d^3x \int dt\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\omega t} \delta\langle\hat{n}(\mathbf{x}t)\rangle \tag{13.16}$$

$$D^R(\mathbf{k},\omega) \equiv \int d^3x \int dt\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\omega t} D^R(\mathbf{x}t) \tag{13.17}$$

Equation (13.14) immediately reduces to

$$\delta\langle\hat{n}(\mathbf{k},\omega)\rangle = \hbar^{-1} D^R(\mathbf{k},\omega)\, e\varphi^{ex}(\mathbf{k},\omega) \tag{13.18}$$

which shows that the system responds at the same wave vector and frequency as the perturbation. This relation is sometimes used to define a generalized susceptibility

$$\chi_{nn}(\mathbf{k},\omega) \equiv \frac{\delta\langle\hat{n}(\mathbf{k},\omega)\rangle}{e\varphi^{ex}(\mathbf{k},\omega)} = \hbar^{-1} D^R(\mathbf{k},\omega) \tag{13.19}$$

Such relations are especially useful in studying transport coefficients, which represent certain long-wavelength and low-frequency limits of the generalized susceptibilities (compare Prob. 9.7).

The foregoing analysis shows that the linear response is most simply expressed in terms of retarded correlation functions of exact Heisenberg operators. Unfortunately, such functions cannot be calculated directly with the Feynman-Dyson perturbation series because Wick's theorem applies only to a time-ordered product of operators. Consequently, it is generally convenient to define an associated time-ordered correlation function of the *same* operators, which necessarily has the form of Eq. (8.8). Wick's theorem can now be used to evaluate the time-ordered correlation function in perturbation theory. The remaining problem of relating the time-ordered and retarded functions can be solved with the Lehmann representation. A specific example has been given in Sec. 7, where $G(\mathbf{k},\omega)$ and $G^R(\mathbf{k},\omega)$ were shown to satisfy Eqs. (7.67) and (7.68). The method is clearly very general, and we state here the corresponding relations for the density correlation functions (Prob. 3.8)

$$\begin{aligned} \operatorname{Re} D(\mathbf{q},\omega) &= \operatorname{Re} D^R(\mathbf{q},\omega) \\ \operatorname{Im} D(\mathbf{q},\omega) \operatorname{sgn}\omega &= \operatorname{Im} D^R(\mathbf{q},\omega) \end{aligned} \tag{13.20}$$

which are valid for real ω. (In this expression, $\operatorname{sgn}\omega \equiv \omega/|\omega|$.) Equations (13.20) are very important, because *any* approximation for $D(\mathbf{q},\omega)$ immediately yields an approximate $D^R(\mathbf{q},\omega)$ and hence the associated linear response. It is also clear from the Lehmann representation for $D(\mathbf{q},\omega)$ that the poles of this

function occur at the exact excitation energies of those states of the interacting assembly that are coupled to the ground state through the density operator.

14□SCREENING IN AN ELECTRON GAS

As our first example, we consider the response of a degenerate electron gas to a static impurity with positive charge Ze, where the external potential is given by

$$\varphi^{ex}(\mathbf{x}t) = Zex^{-1} \tag{14.1}$$

and

$$\varphi^{ex}(\mathbf{q},\omega) = 8\pi^2 Zeq^{-2}\delta(\omega) \tag{14.2}$$

Note that we have here let $t_0 \to -\infty$. This point charge alters the electron distribution in its vicinity, and Eqs. (13.16) and (13.18) together determine the induced particle density to be (for electrons, the interaction is $-e\varphi^{ex}$)

$$\delta\langle\hat{n}(\mathbf{x})\rangle = -(2\pi)^{-3} \int d^3q\, e^{i\mathbf{q}\cdot\mathbf{x}} D^R(\mathbf{q},0)\, 4\pi Ze^2(\hbar q^2)^{-1} \tag{14.3}$$

Equation (12.14) shows that the time-ordered density correlation function D is equal to $\hbar\Pi$, where Π is the time-ordered polarization part. If Π^R is defined as the corresponding retarded polarization, then Eq. (14.3) assumes the simple form

$$\begin{aligned} \delta\langle\hat{n}(\mathbf{x})\rangle &= -(2\pi)^{-3} \int d^3q\, e^{i\mathbf{q}\cdot\mathbf{x}} \Pi^R(\mathbf{q},0)\, 4\pi Ze^2 q^{-2} \\ &= -(2\pi)^{-3} \int d^3q\, e^{i\mathbf{q}\cdot\mathbf{x}} \Pi^R(\mathbf{q},0) ZU_0(\mathbf{q}) \\ &= -(2\pi)^{-3} Z \int d^3q\, e^{i\mathbf{q}\cdot\mathbf{x}} [\Pi^\star(\mathbf{q},0)\, U(\mathbf{q},0)]^R \\ &= -(2\pi)^{-3} Z \int d^3q\, e^{i\mathbf{q}\cdot\mathbf{x}} \{[\kappa^R(\mathbf{q},0)]^{-1} - 1\} \end{aligned} \tag{14.4}$$

where the third line has been obtained with Dyson's equation [see Eqs. (9.43) and (14.5)], and the fourth with the retarded version of Eq. (12.17). The previous perturbation analysis (Sec. 12) allows us to calculate the time-ordered functions Π and κ, and the Lehmann representation then yields [compare Eq. (13.20) for $D \equiv \hbar\Pi$]‡

$$\begin{aligned} \Pi^R(\mathbf{q},\omega) &\equiv (\operatorname{Re} + i \operatorname{sgn}\omega \operatorname{Im})\Pi(\mathbf{q},\omega) \\ &= \operatorname{Re}\Pi(\mathbf{q},\omega) + i\operatorname{sgn}\omega \operatorname{Im}\Pi(\mathbf{q},\omega) \end{aligned} \tag{14.5}$$

$$\kappa^R(\mathbf{q},\omega) = \operatorname{Re}\kappa(\mathbf{q},\omega) + i\operatorname{sgn}\omega\operatorname{Im}\kappa(\mathbf{q},\omega) \tag{14.6}$$

A combination of Eqs. (14.4) and (14.6) then provides an exact description of the screening about a point charge.

In the approximation of retaining only ring diagrams, $\kappa_r(\mathbf{q},0)$ is purely real [Eq. (12.49)], and the retarded function becomes

$$\kappa_r^R(\mathbf{q},0) = \kappa_r(\mathbf{q},0) = 1 + 4\alpha r_s k_F^2(\pi q^2)^{-1} g\left(\frac{q}{k_F}\right) \tag{14.7}$$

‡ Note $[\operatorname{Re} + i\operatorname{sgn}\omega\operatorname{Im}][\operatorname{Re}\kappa + i\operatorname{Im}\kappa]^{-1} = [\operatorname{Re}\kappa - i\operatorname{sgn}\omega\operatorname{Im}\kappa][|\operatorname{Re}\kappa|^2 + |\operatorname{Im}\kappa|^2]^{-1} = [\operatorname{Re}\kappa + i\operatorname{sgn}\omega\operatorname{Im}\kappa]^{-1}$.

Here the function $g(x)$ [Eq. (12.64)] is given by

$$g(x) = \frac{1}{2} - \frac{1}{2x}(1 - \tfrac{1}{4}x^2)\ln\left|\frac{1 - \frac{1}{2}x}{1 + \frac{1}{2}x}\right| \tag{14.8}$$

and has the following limiting behavior

$$g(x) \approx 1 + O(x^2) \qquad x \ll 1 \tag{14.9a}$$

$$g(x) \approx \tfrac{1}{2} + \tfrac{1}{4}(x-2)\ln[\tfrac{1}{4}|x-2|] \qquad |x-2| \ll 1 \tag{14.9b}$$

$$g(x) \sim \tfrac{4}{3}\, x^{-2} \qquad x \gg 1 \tag{14.9c}$$

Equation (14.7) may be substituted into Eq. (14.4); the induced *charge* density then reduces to

$$\begin{aligned} \delta\langle\hat{\rho}(\mathbf{x})\rangle_r &= -e\,\delta\langle\hat{n}(\mathbf{x})\rangle_r \\ &= -Ze\int\frac{d^3q}{(2\pi)^3}\,e^{i\mathbf{q}\cdot\mathbf{x}}\,\frac{4\alpha r_s\pi^{-1}g(q/k_F)}{(q/k_F)^2 + 4\alpha r_s\pi^{-1}g(q/k_F)} \end{aligned} \tag{14.10}$$

This expression has several interesting features:

1. The total induced charge is easily determined as

$$\begin{aligned} \delta Q_r &= \int d^3x\,\delta\langle\hat{\rho}(\mathbf{x})\rangle_r \\ &= -Ze\int d^3q\;\delta(\mathbf{q})\frac{4\alpha r_s\pi^{-1}g(q/k_F)}{(q/k_F)^2 + 4\alpha r_s\pi^{-1}g(q/k_F)} \\ &= -Ze \end{aligned} \tag{14.11}$$

   which shows that the screening is complete at large distances.

2. The integrand of Eq. (14.10) is bounded for all $|q|$ and vanishes like q^{-4} as $q \to \infty$ [compare Eq. (14.9*c*)]. Hence the induced charge density is everywhere finite *including the origin* because

$$\begin{aligned} |\langle\delta\hat{\rho}(\mathbf{x})\rangle_r| &\leq |\langle\delta\hat{\rho}(0)\rangle_r| \\ &= Ze\int\frac{d^3q}{(2\pi)^3}\frac{4\alpha r_s\pi^{-1}g(q/k_F)}{(q/k_F)^2 + 4\alpha r_s\pi^{-1}g(q/k_F)} < \infty \end{aligned} \tag{14.12}$$

   Here the first inequality arises from the oscillatory exponential which reduces the charge density for $x \neq 0$.

3. The singular q^{-2} dependence for small q^2 is cut off at

$$q_{\min} = \left(\frac{4\alpha r_s}{\pi}\right)^{\frac{1}{2}} k_F = \left(\frac{4k_F}{\pi a_0}\right)^{\frac{1}{2}} = \left(\frac{6\pi n e^2}{\epsilon_F^0}\right)^{\frac{1}{2}} \equiv q_{TF} \tag{14.13}$$

   [see Eqs. (3.20 to 3.22), (3.29), and (12.67)], where q_{TF} is the Thomas-Fermi[1]

[1] L. H. Thomas, *Proc. Cambridge Phil. Soc.*, **23**:542 (1927); E. Fermi, *Z. Physik*, **48**:73 (1928). An elementary account of its application to metals may be found in J. M. Ziman, "Principles of the Theory of Solids," secs. 5.1 to 5.3, Cambridge University Press, Cambridge, 1964.

wavenumber. The induced charge density can now be rewritten as

$$\begin{aligned} \delta\langle\hat{\rho}(\mathbf{x})\rangle_r &= -Ze\int\frac{d^3q}{(2\pi)^3}\,e^{i\mathbf{q}\cdot\mathbf{x}}\,\frac{q_{TF}^2\,g(q/k_F)}{q^2 + q_{TF}^2\,g(q/k_F)} \\ &= -\frac{Ze}{2\pi^2 x}\int_0^\infty dq\,q\sin qx\,\frac{q_{TF}^2\,g(q/k_F)}{q^2 + q_{TF}^2\,g(q/k_F)} \end{aligned} \tag{14.14}$$

Since $g(0) = 1$, it is tempting to infer that the induced charge density has the following asymptotic form ($x \to \infty$)

$$\delta\langle\hat{\rho}(\mathbf{x})\rangle_r \sim \delta\rho_{TF}(x) \equiv -Zeq_{TF}^2(4\pi x)^{-1}\,e^{-q_{TF}x} \tag{14.15}$$

where $\delta\rho_{TF}(x)$ is that obtained in the Thomas-Fermi approximation.

The Thomas-Fermi result can be understood very simply in the following way. A noninteracting Fermi gas at $T = 0$ exerts a pressure given by Eq. (5.49b)

$$P = \frac{2}{5}\frac{\hbar^2}{2m}(3\pi^2)^{\frac{2}{3}}\,n^{\frac{5}{3}} \tag{14.16}$$

If we put a charge Ze into a uniform electron gas (imposed on a uniform, positive fixed background of charge density en_0 that makes the unperturbed system neutral), then the condition of local hydrostatic equilibrium requires that the forces on a small (unit) volume element must balance

$$\sum_i \mathbf{F}_i = 0 = -\nabla P - en\mathscr{E} \tag{14.17}$$

where $\mathscr{E}$ is the resulting electric field. Poisson's equation becomes

$$\nabla\cdot\mathscr{E} = -\nabla^2\varphi = 4\pi[Ze\delta(\mathbf{x}) - e(n - n_0)] \tag{14.18}$$

where φ is the electrostatic potential. We can now write

$$n - n_0 = \delta n \tag{14.19a}$$

$$\nabla n = \nabla\delta n \tag{14.19b}$$

and a combination of Eqs. (14.16), (14.17), and (14.19*b*) yields

$$\frac{2}{3}\frac{\hbar^2}{2m}(3\pi^2)^{\frac{2}{3}}\frac{1}{n^{\frac{1}{3}}}\nabla\delta n = e\nabla\varphi \tag{14.20}$$

Since the left side is already linear in small quantities, we can use Eq. (3.29) to write

$$\frac{2}{3}\frac{\hbar^2 k_F^2}{2m}\frac{1}{n_0}\nabla\delta n = e\nabla\varphi \tag{14.21}$$

The divergence of this equation combined with Eq. (14.18) gives

$$(\nabla^2 - q_{TF}^2)\,\delta n(\mathbf{x}) = -Zq_{TF}^2\,\delta(\mathbf{x})$$

or equivalently

$$(\nabla^2 - q_{TF}^2)\,\delta\rho(\mathbf{x}) = Zeq_{TF}^2\,\delta(\mathbf{x}) \tag{14.22}$$

where the Thomas-Fermi wavenumber is defined in Eq. (14.13). The solution to this equation is that quoted in Eq. (14.15)

$$\delta\rho_{TF}(x) = -Zeq_{TF}^2(4\pi x)^{-1}\,e^{-q_{TF}x} \tag{14.23}$$

The approximate result in Eq. (14.15) is *incorrect*, however, because $g(x)$ has a singularity at $x = 2$, where its first derivative becomes infinite. The presence of this singularity in the range of integration $(0 < q < \infty)$ gives $\delta\langle\hat\rho(\mathbf{x})\rangle_r$ an *algebraic* asymptotic dependence on x in contrast to the apparent *exponential* behavior arising from the approximate simple pole at $q = \pm iq_{TF}$. We may extract the correct asymptotic behavior of $\delta\langle\hat\rho(\mathbf{x})\rangle_r$ in the following manner: first rewrite the logarithm appearing in $g(q/k_F)$ as [see Eq. (14.8)]

$$\ln\left|\frac{q-2k_F}{q+2k_F}\right| = \lim_{\eta\to 0}\tfrac{1}{2}\ln\frac{(q-2k_F)^2+\eta^2}{(q+2k_F)^2+\eta^2}$$

Since g is an even function of its argument, the integral in Eq. (14.14) can be written as

$$\delta\langle\hat\rho(\mathbf{x})\rangle_r = \frac{Ze}{4\pi^2 ix}\int_{-\infty}^{\infty} q\,dq\,e^{iqx}\left[\frac{q^2}{q^2+q_{TF}^2 g(q/k_F)} - 1\right] \tag{14.24}$$

The integrand is now an analytic function of q with the singularity structure shown in Fig. 14.1, and the branch cuts of the logarithms have been chosen so that the logarithm is real along the real axis. The contour can be deformed as indicated, and the pole at $q \approx iq_{TF}$ gives the contribution of Eq. (14.15), which vanishes exponentially for large x. In contrast, the cuts extend down to (within η) the real axis. The integrals along the two branch cuts depend on the difference of the function across the cut; this difference arises solely from the phase of the logarithm, and with the branch cuts as shown we have

$$\Delta\left[\frac{1}{2}\log\frac{(q-2k_F)^2+\eta^2}{(q+2k_F)^2+\eta^2}\right] = \begin{cases} \pi & \text{on } C_1 \\ -\pi & \text{on } C_2 \end{cases}$$

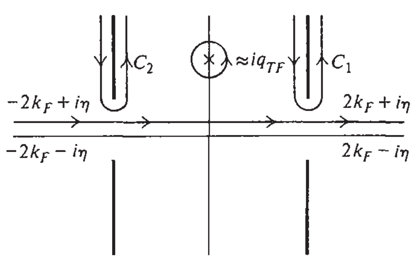

Fig. 14.1 Contour for asymptotic evaluation of $\delta\langle\hat\rho(\mathbf{x})\rangle_r$.

where Δ indicates the value of the phase on the right side of the cut minus the value of the phase on the left. Because of the decreasing exponential in the integrand, all the slowly varying functions in the integrand can then be replaced by their values at the start of the branch cut. Thus we have

$$\begin{aligned}\delta\langle\hat\rho(\mathbf{x})\rangle_r &\underset{x\to\infty}{\sim} \frac{Ze}{4\pi^2 ix}\lim_{\eta\to 0}\left(\int_{C_1} + \int_{C_2}\right) q\,dq\,e^{iqx}\left(q^2\left\{q^2 + q_{TF}^2\left[\frac{1}{2} - \frac{k_F}{2q}\right.\right.\right.\\ &\qquad\left.\left.\left.\times\left(1 - \frac{q^2}{4k_F^2}\right)\frac{1}{2}\log\frac{(q-2k_F)^2+\eta^2}{(q+2k_F)^2+\eta^2}\right]\right\}^{-1} - 1\right)\\ &\underset{x\to\infty}{\sim} \frac{Ze}{4\pi^2 ix}\lim_{\eta\to 0}\left[\frac{\pi q_{TF}^2}{4k_F}\frac{8k_F^3}{(4k_F^2 + \frac{1}{2}q_{TF}^2)^2}\right.\\ &\qquad\left.\times\left(e^{-2ik_Fx}\,i\int_\eta^\infty ue^{-ux}\,du + e^{2ik_Fx}\,i\int_\eta^\infty ve^{-vx}\,dv\right)\right]\end{aligned} \tag{14.25}$$

where we have introduced $q = 2k_F + iv$ along C_1 and $q = -2k_F + iu$ along C_2. The remaining integrals are elementary, and we find

$$\delta\langle\hat\rho(\mathbf{x})\rangle_r \underset{x\to\infty}{\sim} \frac{Ze}{\pi}\frac{2\xi}{(4+\xi)^2}\frac{\cos(2k_Fx)}{x^3} \tag{14.26a}$$

$$\xi \equiv \frac{q_{TF}^2}{2k_F^2} \tag{14.26b}$$

which was first derived by Langer and Vosko.[1] The expression (14.26*a*) is qualitatively different from that predicted in Eq. (14.15) and exhibits long-range oscillations with a radial wavelength π/k_F and an envelope proportional to x^{-3}. It is clear that $\delta\langle\hat\rho(\mathbf{x})\rangle_r$ is an improvement over $\delta\rho_{TF}(\mathbf{x})$, since the former incorporates the distribution function of the interacting medium in computing the response to the external field.

From a physical point of view, the long-range oscillations in the screening charge arise from the sharp Fermi surface, because it is not possible to construct a smooth function out of the restricted set of wave vectors $q > k_F$. This effect was first suggested by Friedel,[2] and such *Friedel oscillations* have been observed as a broadening of nuclear magnetic resonance lines in dilute alloys.[3] A similar effect also occurs in dilute magnetic alloys; the conduction electrons induce an indirect interaction between magnetic impurities of the form $x_{ij}^{-3}\cos(2k_Fx_{ij})$, where x_{ij} is the separation of the impurities.[4] At low but finite temperatures, the Fermi surface is smeared over a thickness k_BT in energy, and it turns out that

[1] J. S. Langer and S. H. Vosko, *J. Phys. Chem. Solids*, **12**:196 (1960).
[2] J. Friedel, *Phil. Mag.*, **43**:153 (1952); *Nuovo Cimento*, **7**:287, *Suppl.* 2 (1958).
[3] N. Bloembergen and T. J. Rowland, *Acta Met.*, **1**:731 (1953); T. J. Rowland, *Phys. Rev.*, **119**:900 (1960); W. Kohn and S. H. Vosko, *Phys. Rev.*, **119**:912 (1960); see also, J. M. Ziman, *op. cit.*, secs. 5.4 and 5.5.
[4] M. A. Ruderman and C. Kittel, *Phys. Rev.*, **96**:99 (1954).

Eq. (14.26*a*) must be multiplied by the factor $\exp(-2\pi m k_B T x/\hbar^2 k_F)$. The importance of a sharp Fermi surface is confirmed by the behavior in a superconductor, where the Fermi surface is smeared over an energy width $\Delta \ll \epsilon_F^0$ even at $T = 0$ (see Chap. 13). In this case, the asymptotic form of the screening density is proportional to $x^{-3}\cos(2k_F x)\exp(-k_F x\Delta/\epsilon_F^0)$, completely analogous to that for a normal metal at finite temperature.[1]

15□PLASMA OSCILLATIONS IN AN ELECTRON GAS

It has already been pointed out that $\Pi(\mathbf{q},\omega)$ has poles at the exact excitation energy of those collective states of the interacting system that are connected to the ground state through the density operator. Recalling Eqs. (9.43*a*) and (9.46)

$$\frac{U(q)}{U_0(\mathbf{q})} = \frac{1}{\kappa(\mathbf{q},\omega)} = 1 + U_0(\mathbf{q})\Pi(\mathbf{q},\omega) \tag{15.1}$$

we observe that $\kappa(\mathbf{q},\omega)$ vanishes at these same energies. In the ring approximation, Eq. (12.50) shows that the dielectric constant κ_r has one obvious zero, occurring for fixed energy transfer ν and long wavelengths $q \to 0$

$$\kappa_r(\mathbf{q},\omega) = 1 - \frac{4\alpha r_s}{3\pi\nu^2} \tag{15.2}$$

This quantity vanishes at

$$\nu_{pl}^2 = 4\alpha r_s(3\pi)^{-1} \tag{15.3}$$

Rewriting this expression in dimensional units [see Eqs. (3.20) to (3.22), (12.35), and (12.52)] we find a collective excitation at the classical plasma frequency[2] given by

$$\Omega_{pl}^2 = \frac{4\pi n e^2}{m} \tag{15.4}$$

We shall investigate these plasma oscillations in more detail by considering the linear response of a degenerate electron gas to an impulsive perturbation

$$\varphi^{ex}(\mathbf{x}t) = e^{i\mathbf{q}\cdot\mathbf{x}}\varphi_0\,\delta(t) \tag{15.5}$$

whose Fourier transform is given by

$$\varphi^{ex}(\mathbf{k},\omega) = \varphi_0(2\pi)^3\,\delta(\mathbf{q}-\mathbf{k}) \tag{15.6}$$

The corresponding induced density perturbation becomes

$$\begin{aligned}\delta\langle\hat{n}(\mathbf{x}t)\rangle &= -e(2\pi)^{-4}\int d^3k\,d\omega\,e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i\omega t}\,\Pi^R(\mathbf{k},\omega)\,\varphi^{ex}(\mathbf{k},\omega)\\ &= -e\varphi_0\,e^{i\mathbf{q}\cdot\mathbf{x}}(2\pi)^{-1}\int d\omega\,e^{-i\omega t}\,\Pi^R(\mathbf{q},\omega)\\ &= -e\varphi_0\,e^{i\mathbf{q}\cdot\mathbf{x}}(2\pi)^{-1}\int d\omega\,e^{-i\omega t}\,U_0(\mathbf{q})^{-1}\{[\kappa^R(\mathbf{q},\omega)]^{-1}-1\}\end{aligned} \tag{15.7}$$

[1] A. L. Fetter, *Phys. Rev.*, **140:**A1921 (1965).

[2] The classical theory of plasma oscillations is discussed at the end of this section.

which shows that the singularities of Π^R in the complex ω plane also determine the resonant frequencies of the system.

Although Eq. (15.7) is exact, we shall consider only the approximation of retaining the ring diagrams. In this case $\kappa_r^R(\mathbf{q},\omega)$ is given by Eqs. (12.24) and (14.6) as

$$\kappa_r^R(\mathbf{q},\omega) = 1 - V(\mathbf{q})\Pi^{0R}(\mathbf{q},\omega) \tag{15.8}$$

where [compare Eqs. (12.29) and Eqs. (14.5)]

$$\begin{aligned}\Pi^{0R}(\mathbf{q},\omega) &= \operatorname{Re}\Pi^0(\mathbf{q},\omega) + i\operatorname{sgn}\omega\operatorname{Im}\Pi^0(\mathbf{q},\omega)\\ &= \frac{2}{\hbar}\int\frac{d^3k}{(2\pi)^3}\left[\frac{(1-n^0_{\mathbf{k}+\mathbf{q}})n^0_{\mathbf{k}}}{\omega+\omega_{\mathbf{k}}-\omega_{\mathbf{q}+\mathbf{k}}+i\eta} - \frac{n^0_{\mathbf{k}+\mathbf{q}}(1-n^0_{\mathbf{k}})}{\omega+\omega_{\mathbf{k}}-\omega_{\mathbf{k}+\mathbf{q}}+i\eta}\right]\\ &= -\frac{2}{\hbar}\int\frac{d^3k}{(2\pi)^3}\left[\frac{n^0_{\mathbf{k}+\mathbf{q}}-n^0_{\mathbf{k}}}{\omega-(\omega_{\mathbf{k}+\mathbf{q}}-\omega_{\mathbf{k}})+i\eta}\right]\end{aligned} \tag{15.9}$$

where $n^0_{\mathbf{k}} = \theta(k_F - k)$. Thus Π^{0R} differs from Π^0 only in the infinitesimals $\pm i\eta$. The frequency and lifetime of the collective modes are determined by the poles of the integrand in Eq. (15.7).[1] These occur at the values $\Omega_q - i\gamma_q$ that satisfy the equation

$$1 = V(\mathbf{q})\Pi^{0R}(\mathbf{q},\Omega_q - i\gamma_q) \tag{15.10}$$

In general, this equation can be solved only with numerical analysis; if the damping is small ($\gamma_q \ll \Omega_q$), however, then the real and imaginary parts separate, and we find

$$1 = V(\mathbf{q})\operatorname{Re}\Pi^{0R}(\mathbf{q},\Omega_q) = V(\mathbf{q})\operatorname{Re}\Pi^0(\mathbf{q},\Omega_q) \tag{15.11}$$

$$\begin{aligned}\gamma_q &= \operatorname{Im}\Pi^{0R}(\mathbf{q},\Omega_q)\left[\frac{\partial\operatorname{Re}\Pi^{0R}(\mathbf{q},\omega)}{\partial\omega}\bigg|_{\Omega_q}\right]^{-1}\\ &= \operatorname{sgn}\Omega_q\operatorname{Im}\Pi^0(\mathbf{q},\Omega_q)\left[\frac{\partial\operatorname{Re}\Pi^0(\mathbf{q},\omega)}{\partial\omega}\bigg|_{\Omega_q}\right]^{-1}\end{aligned} \tag{15.12}$$

Equation (15.11) determines the dispersion relation Ω_q of the collective mode, while Eq. (15.12) then yields an explicit formula for the damping constant. This approximate separation of real and imaginary parts will be shown to be valid at long wavelengths, and we now consider the expansion of Π^{0R} for $q \to 0$.

Although it is possible to expand Eq. (12.36) for small q, we instead work directly with Eq. (15.9). A simple change of variables in the first term of this

[1] In general, Π^R also has a cut in the complex ω plane just below the real axis, with a discontinuity proportional to $\operatorname{Im}\Pi^R(\mathbf{q},\omega)$ (see, for example, Fig. 12.9). As $t \to \infty$, however, this cut makes a negligible contribution to Eq. (15.7); hence the dominant long-time behavior here arises from the collective mode, which is undamped in the present approximation.