Electrons as scattered waves

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- Experiment

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Electrons as plane waves

Electron in vacuum is a plane wave

$$\psi_{\vec{k}}(\vec{r},t) = \frac{1}{\sqrt{V}} \exp(i\vec{k}\vec{r} - iE(k)t/\hbar)$$

$\psi_{\vec{k}}(\vec{r},t)$ - wavefunction

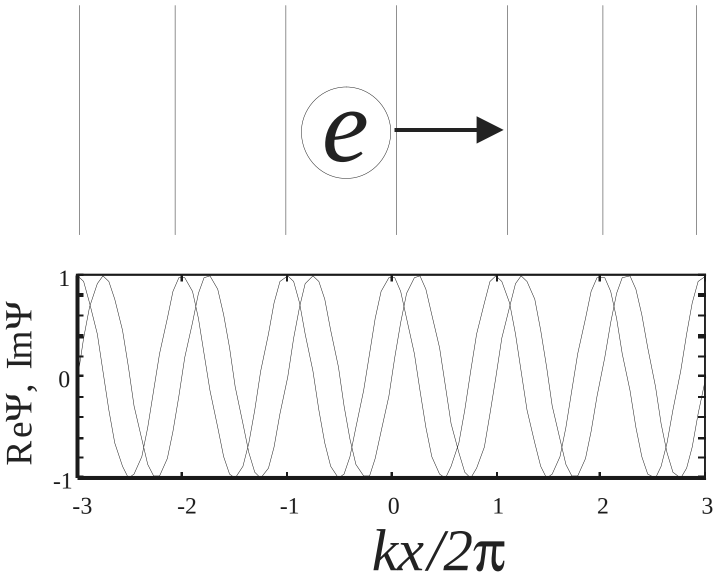
$\left|\psi_{\vec{k}}(\vec{r},t)\right|^2$ - probability

V - norm. volume

$\vec{k}$ - wavevector

$\vec{p} = \vec{k}\hbar$ - momentum

$E = \frac{(\vec{k}\hbar)^2}{2m}$ - energy



Electrons as fermions

Pauli principle: A state is either filled or empty

Number of states in a small cube near **k**

$$2_s \frac{dk_x dk_y dk_z}{(2\pi)^3} V$$

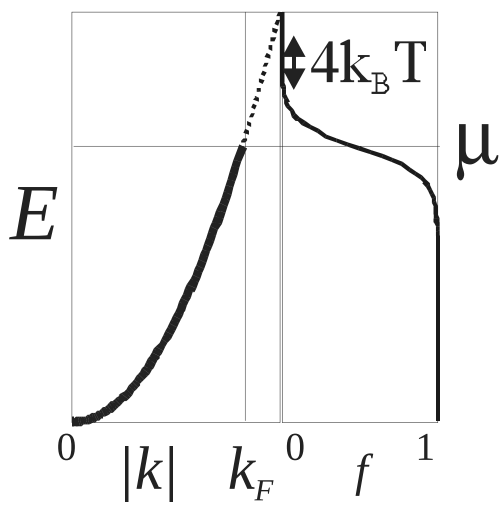
Fraction of filled states in the cube: filling factor ***f***

$$\begin{bmatrix} density \\ energy\ density \\ current\ density \end{bmatrix} = \begin{bmatrix} \rho \\ \mathrm{E} \\ \vec{j} \end{bmatrix} = \int 2_s \frac{d^3\vec{k}}{(2\pi)^3} \begin{bmatrix} 1 \\ E(\vec{k}) \\ e\vec{v}(\vec{k}) \end{bmatrix} f(\vec{k})$$

q.mech. does not set *f*, statistics does

$$f_{eq}(\vec{k}) = f_F(E(\vec{k}) - \mu)) = \frac{1}{1 + \exp\frac{E-\mu}{k_B T}}$$

Zero T: sharp step



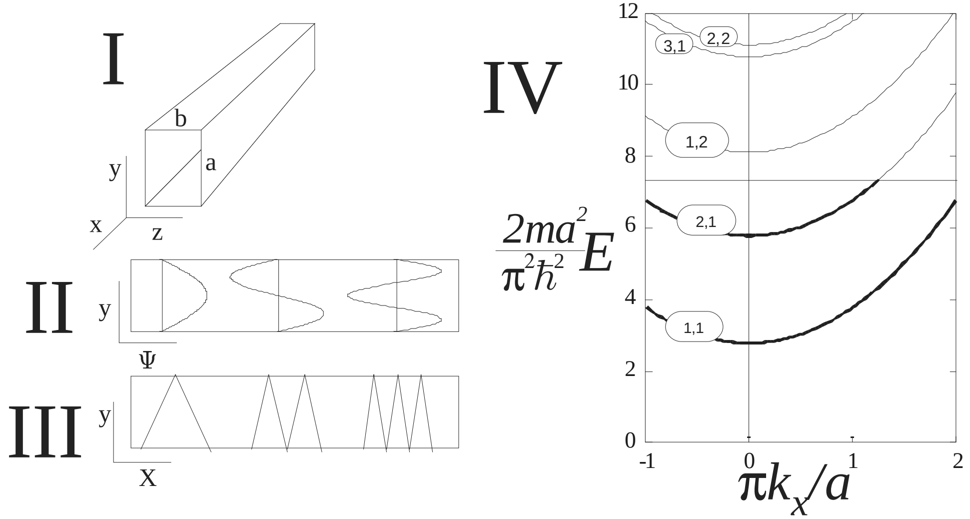
Waveguide

1d motion: y,z – restricted: mode index n, x- free: k_x

$$\psi_{k_x,n}(x,y,z) = \Phi_n(y,z)\exp(ik_x x)$$

Standing wave

Plane wave

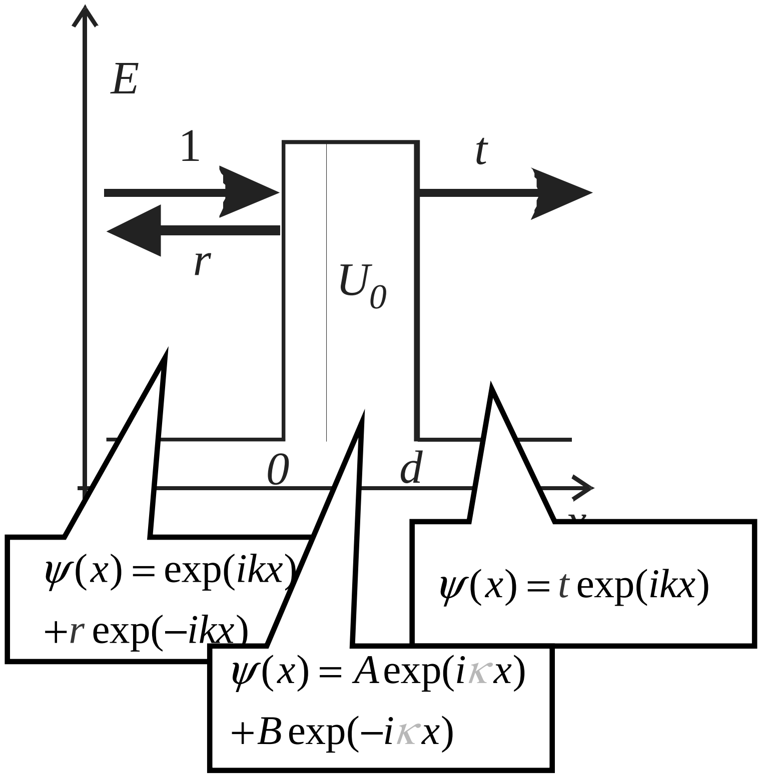


$$E_n(k_x) = \frac{(\hbar k_x)^2}{2m} + E_n;\ E_n = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_y^2}{a^2} + \frac{n_z^2}{b^2}\right)$$

Dictionary: mode = transport channel

Potential barrier

Intersect the waveguide with a potential barrier U(x)



$$\frac{(\hbar k)^2}{2m} = E; \frac{(\hbar \kappa)^2}{2m} = E - U_0$$

4 unknown variables:

A,B,r,t

4 equations:

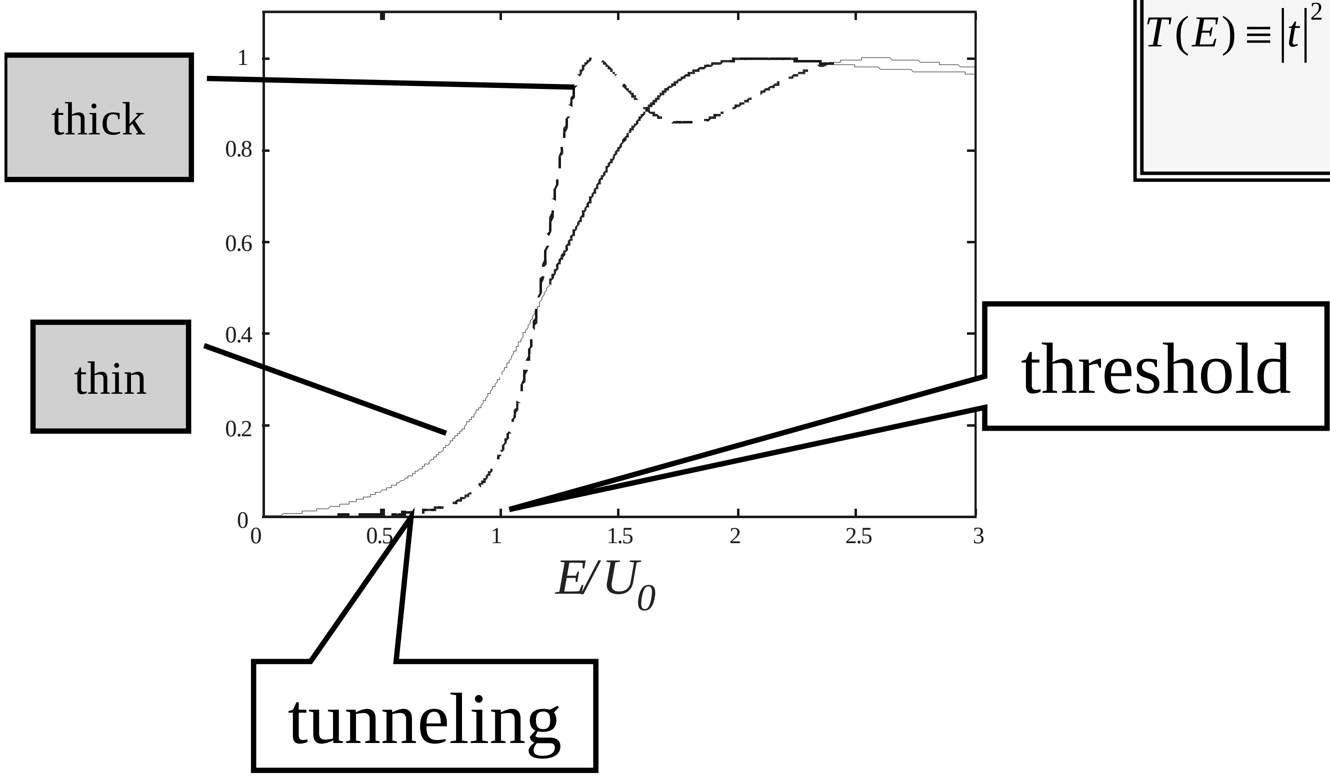
continuity of wavefunction and its derivative at two boundaries

Dictionary: reflection and transmission amplitudes

Potential barrier: transmission probability

Result for a rectangular barrier

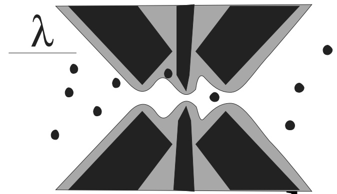
$$T(E) \equiv |t|^2 = \frac{1}{1+\left(\frac{\sin \kappa d\,(k^2-\kappa^2)}{2k\kappa}\right)^2}$$



classical mechanics:
$T(E) = 0 \text{ or } 1$

Dictionary: transmission probability, coefficient

Nanostructure versus waveguide

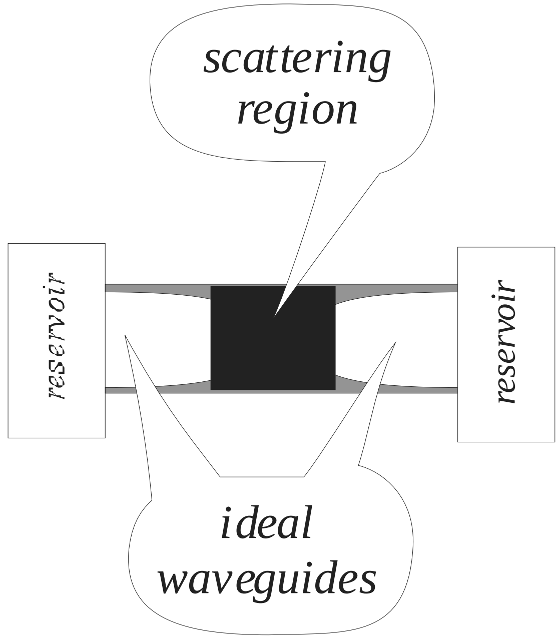


Nanostructure: can be very complex

Can be modelled as: waveguide with transport channels + potential barrier

Essence: set of transmissions T_n

Enough to describe the transport!



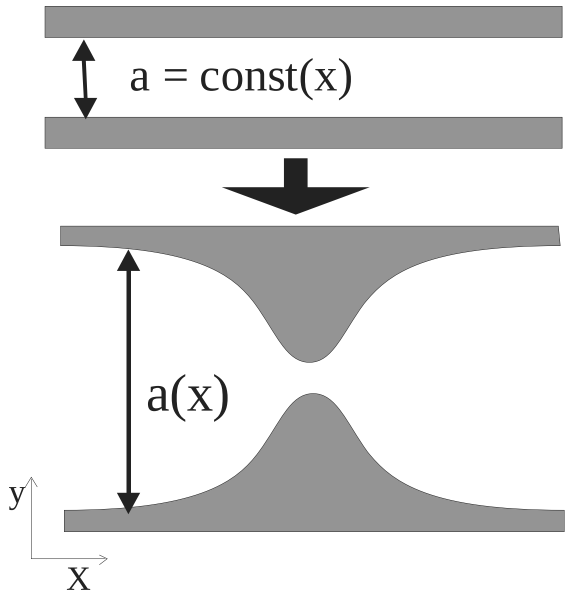
To see this: consider

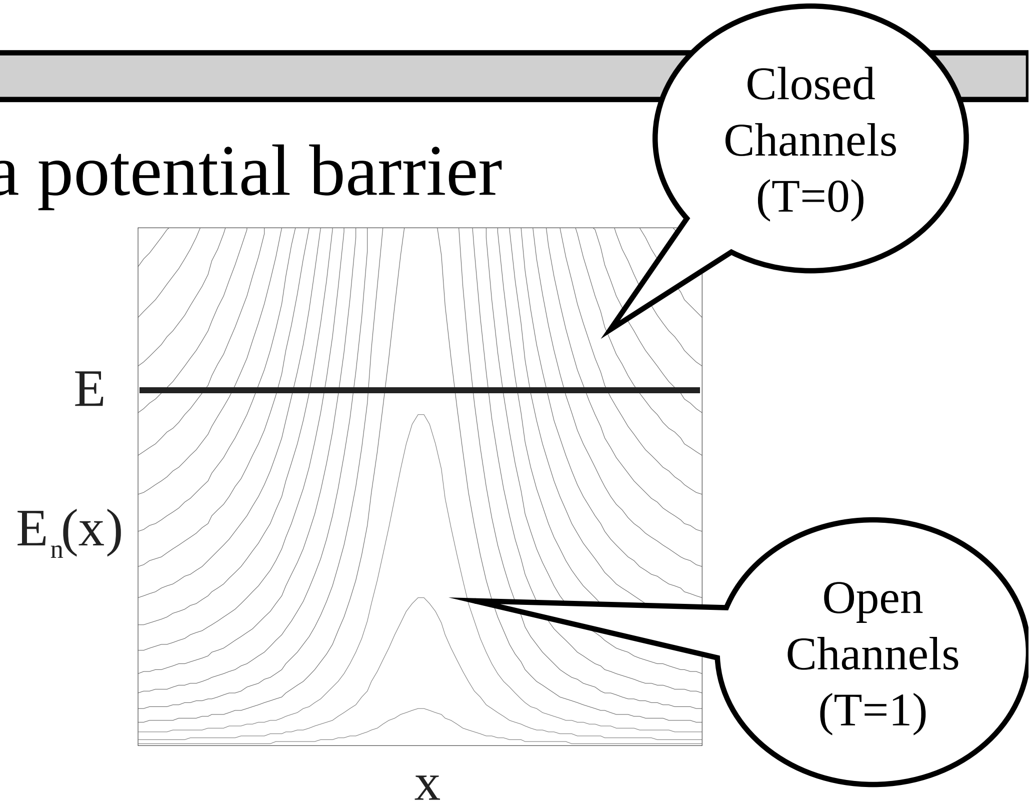
Adiabatic Quantum Transport,

Quantum Point Contact

Adiabatic Quantum Transport

Constriction as a potential barrier

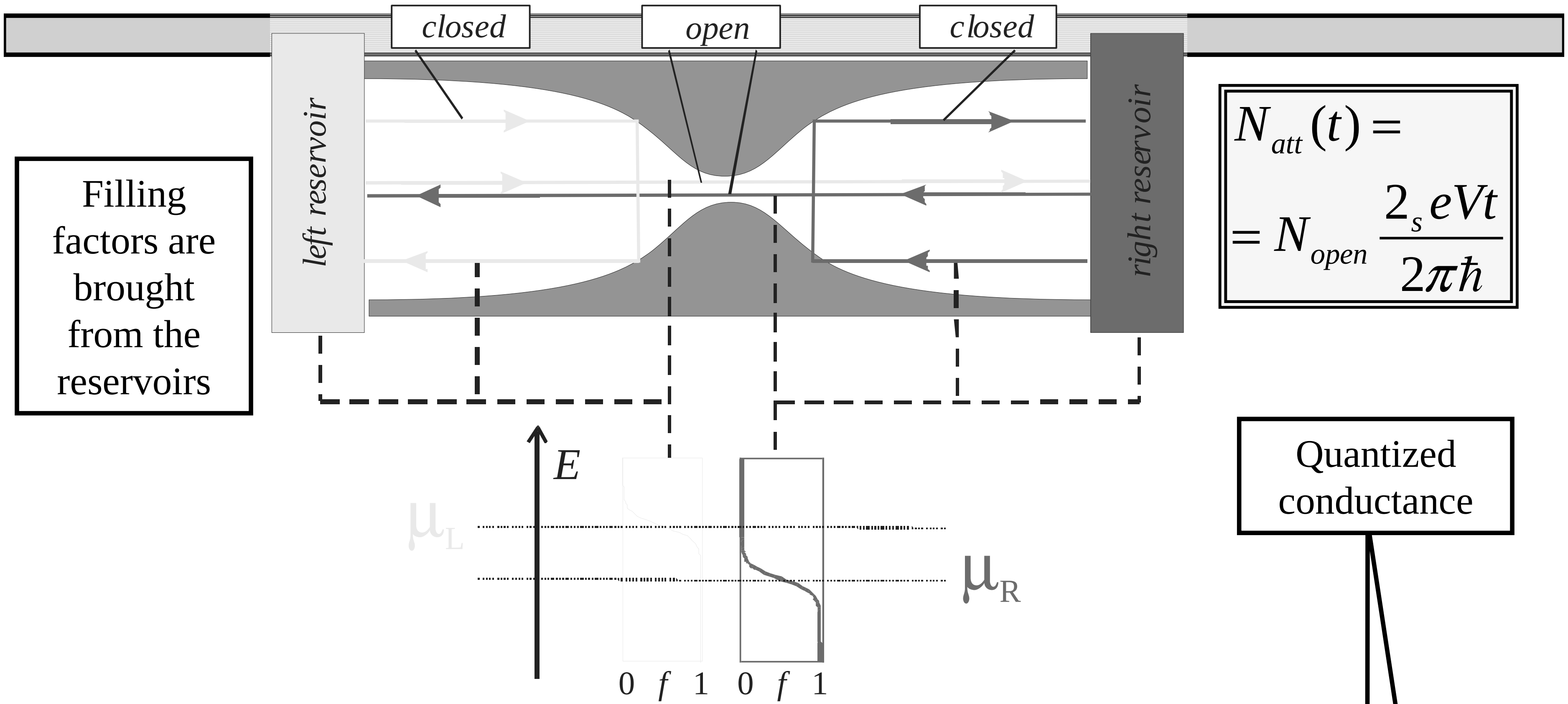




$$E_n(x) = \frac{\pi^2 \hbar^2}{2m}\left(\frac{n_y^2}{a^2(x)} + \frac{n_y^2}{b^2(x)}\right)$$

Current in a QPC

Filling factors are brought from the reservoirs



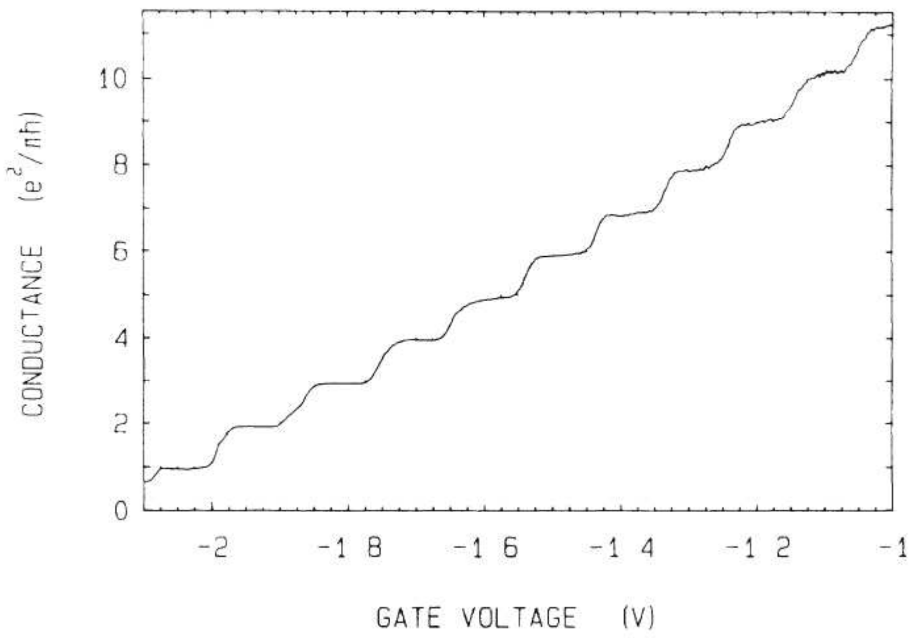
$$N_{att}(t) = N_{open}\frac{2_s eVt}{2\pi\hbar}$$

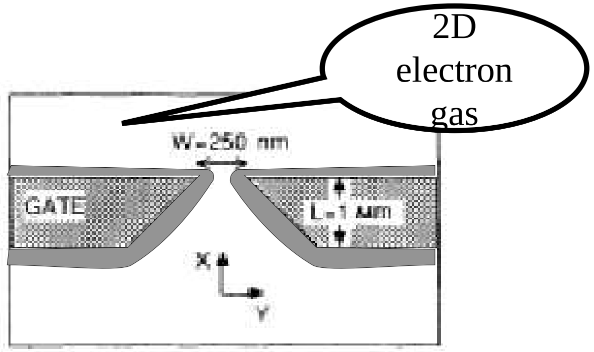
Quantized conductance

$$I = \frac{2_s e}{2\pi\hbar}\sum_{n:open}\int dE\left[f_L(E) - f_R(E)\right] = \frac{2_s e}{2\pi\hbar}N_{open}(\mu_L - \mu_R) = G_Q N_{open} V$$

Experiment

Van Wees et al. 1988 –experimental evidence of conductance quantization



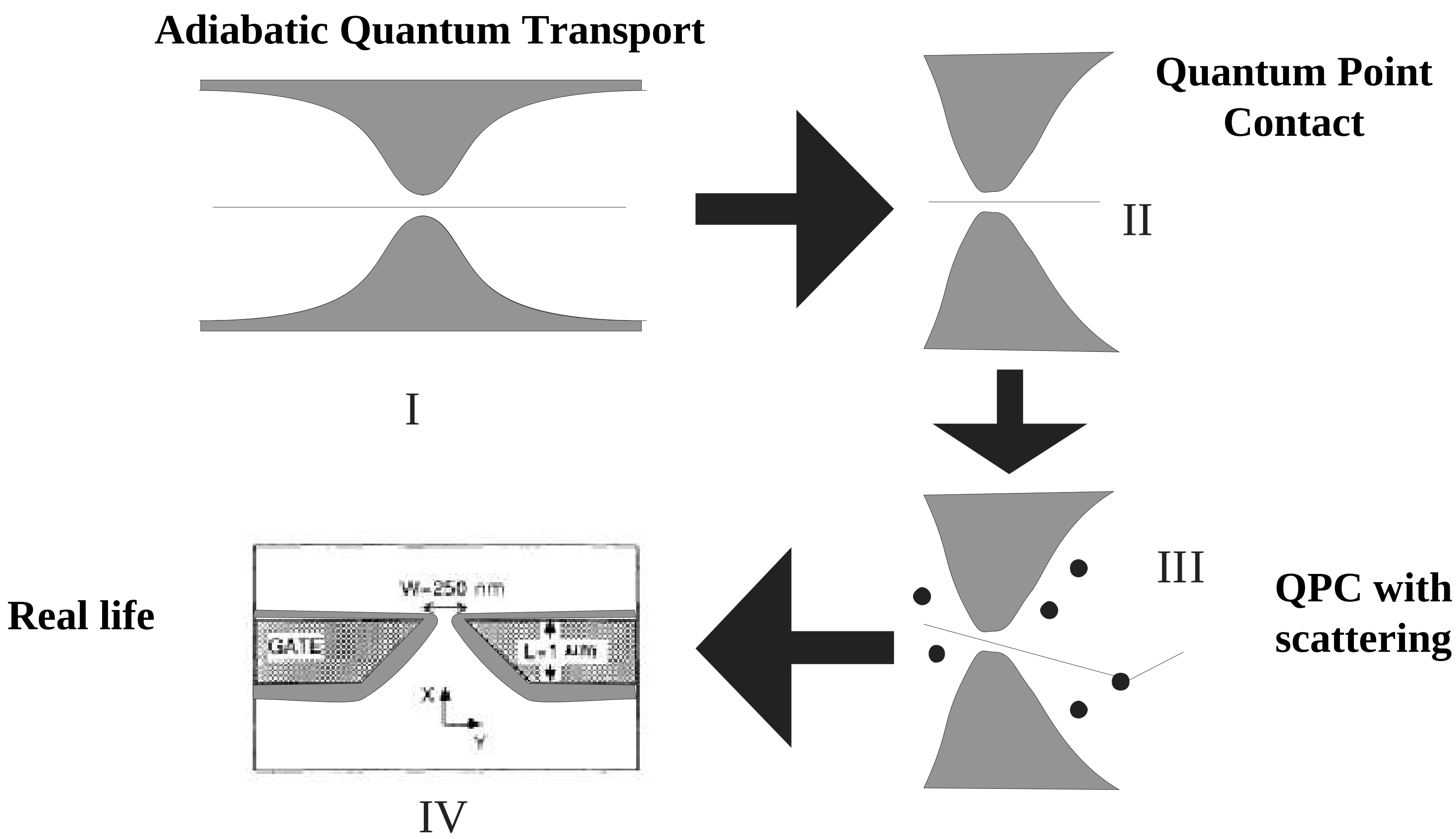


Infinitely many transport channels –

Few open channels

Quantization not ideal

Building a Landauer conductor



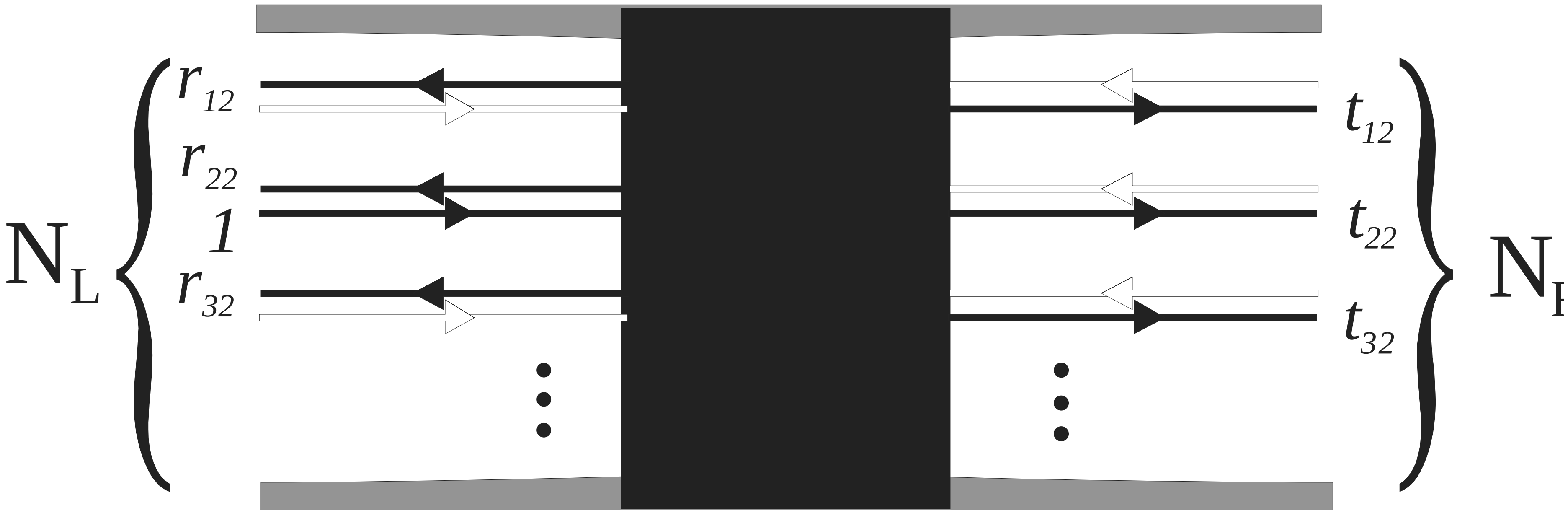
Scattering matrix

$N_R + N_L$ Incoming amplitudes $\vec{a}$

$N_R + N_L$ Outgoing amplitudes $\vec{b}$

$$\vec{b} = \hat{s}\vec{a}$$

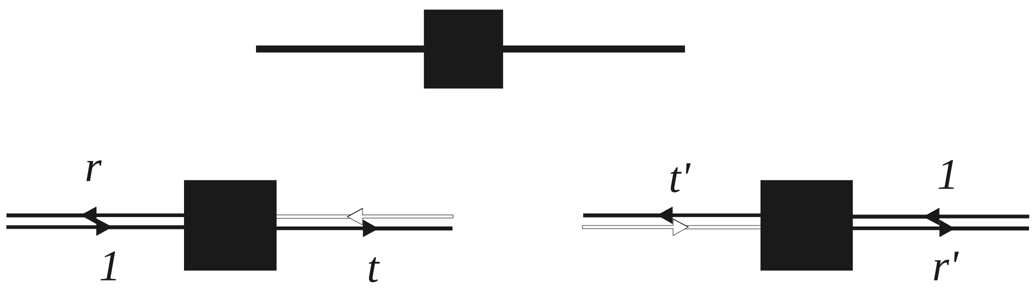
$$\begin{bmatrix} \vec{b}_L \\ \vec{b}_R \end{bmatrix} = \begin{bmatrix} \hat{r} & \hat{t}' \\ \hat{t} & \hat{r}' \end{bmatrix} \begin{bmatrix} \vec{a}_L \\ \vec{a}_R \end{bmatrix}$$



Scattering matrix:properties and example

$$\hat{s}^{+}\hat{s}=\hat{1}$$ unitarity $\Rightarrow$ $$\hat{r}^{+}\hat{r}+\hat{t}^{+}\hat{t}=\hat{1}$$

$$\hat{t}=\hat{t}\,'$$ Time reversability



One-channel scatterer

$$\begin{bmatrix} b_L \\ b_R \end{bmatrix} = \begin{bmatrix} r & t' \\ t & r' \end{bmatrix} \begin{bmatrix} a_L \\ a_R \end{bmatrix}$$

$$r=\sqrt{R}e^{i\theta};\ r'=-\sqrt{R}e^{i(2\eta-\theta)};$$

$$t=\sqrt{T}e^{i\eta};\ T=1-R$$

Landauer formula

Hermitian matrix $\hat{t}^{+}\hat{t}$ **Has a set of eigenvalues: transmissions** T_p At each energy *E*

The current reads:

$$I = G_Q \sum_p \int dE\, T_p(E)\,(f_L(E) - f_R(E))$$

$$I = G_Q \sum_p T_p \; V$$

Simple-minded derivation

One channel:

Reservoir biased at voltage *V* sends $N_{att}(t) = \frac{2_s eVt}{2\pi\hbar}$ **electrons**

Chance to pass: T_0 Charge passed: Q = e $T_0 N_{att}$

Average current: $I = Q/t = G_Q\, T_0\, V$

Many channels: sum over channels

$$I = G_Q \sum_p T_p \; V$$

Restrictions and limitations

Restrictions:

•Elastic scattering: electrons pass without energy loss

•Electrons do not interact (*the same?)*

Limitations:

•Nature is generally merciful

•Electrons do not interact close to Fermi level

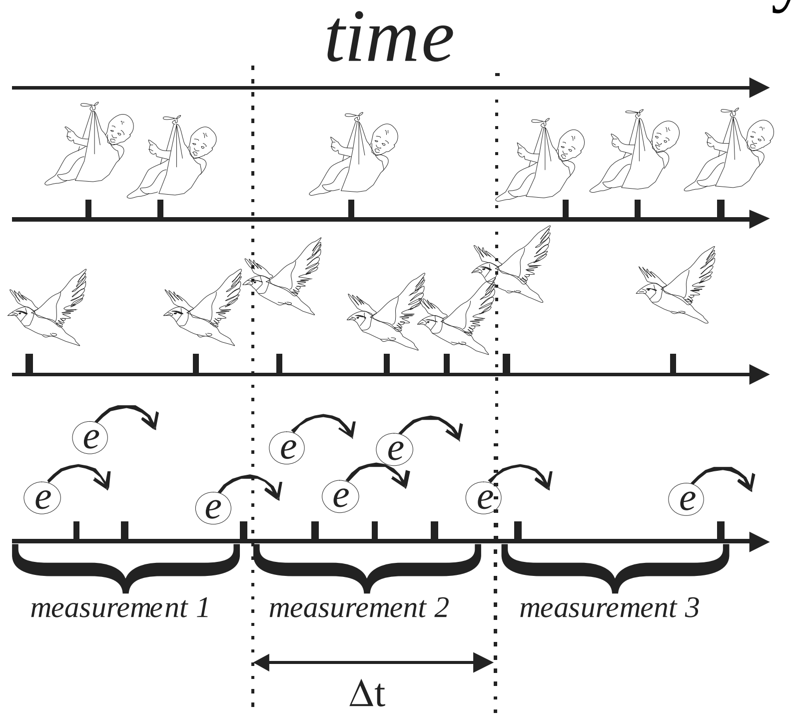
• Low temperature, voltage are good

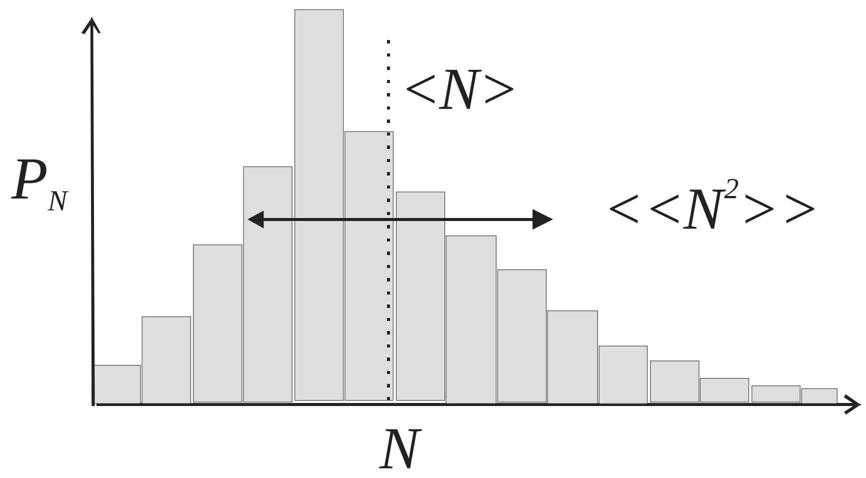
• Short structures are good

• Limitations depend on quantity of interest

Counting electrons

How to count correctly?





How to count electrons?

Characteristic function

$$\Lambda_t(\chi) = \sum_N \exp(i\chi N) P_t(N)$$

- Indepentent events: Λ factorizes
- Time property: $\ln \Lambda_t(\chi) \propto t$
- Elementary event analysis

single

Couple

$$\ln \Lambda_t(\chi) = t \ln(1 + W_1(\exp(i\chi) - 1) + W_2(\exp(i2\chi) - 1) + \ldots$$
$$+ W_M(\exp(iM\chi) - 1))$$

Flock

Two limits

Tunnel junction: $T_n << 1$: rare=independent electron transfers

$$\Lambda_t(\chi) = \exp(\tilde{N}(e^{i\chi} - 1)); \quad \tilde{N} \equiv t < I > / e$$

QPC: T_n=1: electrons are waves: current does not fluctuate

$$\Lambda_t(\chi) = \exp(i\chi\tilde{N}); \quad \tilde{N} \equiv t < I > / e$$

Levitov formula

$$\ln \Lambda(\chi) = 2_s \Delta t \int \frac{dE}{2\pi\hbar} \sum_p \ln \left\{ \begin{array}{l} 1 + T_p (e^{i\chi} - 1) f_L (1 - f_R) + \\ T_p (e^{-i\chi} - 1) f_R (1 - f_L) \end{array} \right\}$$

Electron transfers

- Independent in different energy strip
- Independent in different channels
- To the left and to the right: are dependent!
- Transfers at negative energy are blocked!

blocking factor

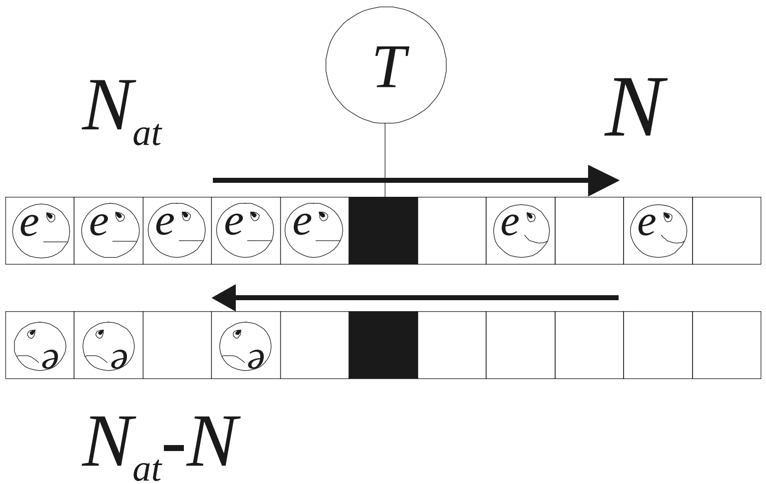
$f_L (1 - f_R)$

Simple example: electrons transferred in one direction

Electrons gambling

$$\ln \Lambda(\chi) = N_{at} \ln\left\{1 + T_0(e^{i\chi} - 1)\right\}$$

$$N_{at} = \frac{2_s eVt}{2\pi\hbar}$$



N_{at} = number of game slots

T_0 = winning chance

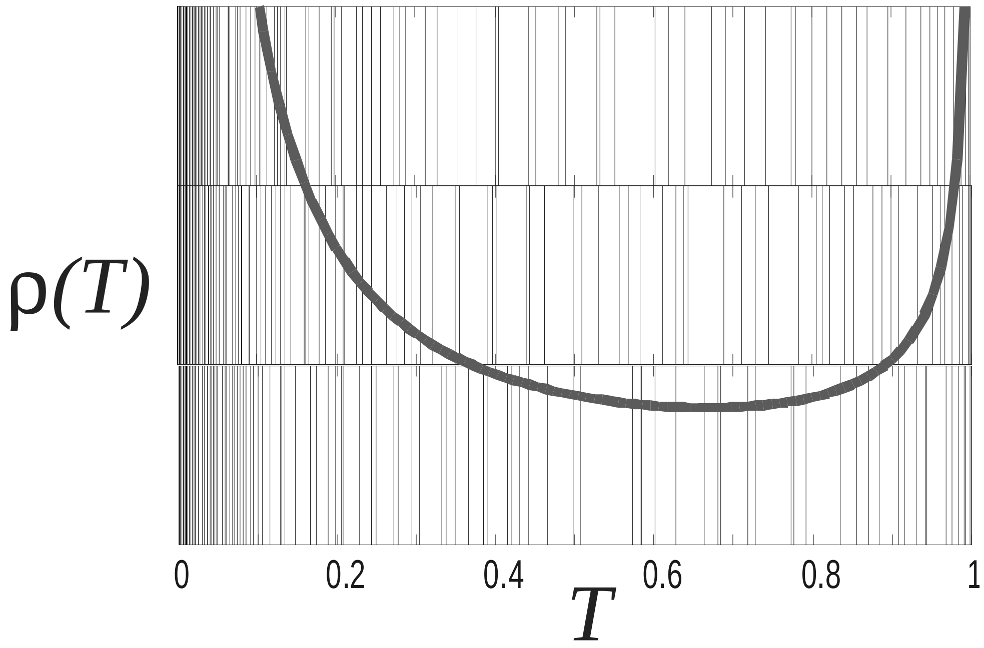
N = number of games won

$$P_N = \binom{N_{at}}{N} T_0^N (1 - T_0)^{N_{at} - N}$$

Transmission distribution

Person ⇒ pin-code ⇐ ATM machine

Quantum Contact ⇒ T_p ⇐ Quantum Transport



$$\rho(T) = \left\langle \sum_p \delta(T - T_p) \right\rangle$$

Diffusive contact:

$$\rho(T) = \frac{2\langle G \rangle}{G_Q} \frac{1}{T\sqrt{1-T}}$$

Types of quantum contacts

Quantum contact: an individuality: pin-code $\boldsymbol{T_p}$

Type of a quantum contact:= shape of transmission distribution

Tunnel junction: $T_p \approx 0$ Realistic QPC $T_p \approx 1$