Appendix A

Fourier transformations

Fourier transformation is useful to employ in the case of homogeneous systems or to change linear differential equations into linear algebraic equations. The idea is to resolve the quantity $f(\mathbf{r}, t)$ under study on plane wave components,

$$f_{\mathbf{k},\omega}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{A.1}$$

travelling at the speed $v = \omega/|\mathbf{k}|$.

A.1 Continuous functions in a finite region

Consider a rectangular box in 3D with side lengths L_x, L_y, L_z and a volume $\mathcal{V} = L_x L_y L_z$. The central theorem in Fourier analysis states that any well-behaved function fulfilling the periodic boundary conditions,

$$f(\mathbf{r} + L_x\mathbf{e}_x) = f(\mathbf{r} + L_y\mathbf{e}_y) = f(\mathbf{r} + L_z\mathbf{e}_z) = f(\mathbf{r}) \tag{A.2}$$

can be written as a Fourier series

$$f(\mathbf{r}) = \frac{1}{\mathcal{V}}\sum_{\mathbf{k}} f_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \begin{cases} k_x = \frac{2\pi n_x}{L_x}, \ n_x = 0, \pm1, \pm2, \ldots \\ \text{likewise for } y \text{ and } z, \end{cases} \tag{A.3}$$

where

$$f_{\mathbf{k}} = \int_{\mathcal{V}} d\mathbf{r}\, f(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{A.4}$$

Note the prefactor $1/\mathcal{V}$ in Eq. (A.3). It is our choice to put it there. Another choice would be to put it in Eq. (A.4), or to put $1/\sqrt{\mathcal{V}}$ in front of both equations. In all cases the product of the normalization constants should be $1/\mathcal{V}$.

An extremely important and very useful theorem states

$$\int d\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}} = \mathcal{V}\,\delta_{\mathbf{k},0}, \qquad \frac{1}{\mathcal{V}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} = \delta(\mathbf{r}). \tag{A.5}$$

Note the dimensions in these two expressions so that you do not forget where to put the factors of $\mathcal{V}$ and $1/\mathcal{V}$. Note also that by using Eq. (A.5) you can prove that Fourier transforming from $\mathbf{r}$ to $\mathbf{k}$ and then back brings you back to the starting point: insert $f_\mathbf{k}$ from Eq. (A.4) into the expression for $f(\mathbf{r})$ in Eq. (A.3) an reduce by use of Eq. (A.5).

A.2 Continuous functions in an infinite region

If we let $\mathcal{V}$ tend to infinity the $\mathbf{k}$-vectors become quasi-continuous variables, and the $\mathbf{k}$-sum in Eq. (A.3) is converted into an integral,

$$f(\mathbf{r}) = \frac{1}{\mathcal{V}} \sum_\mathbf{k} f_\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}} \underset{\mathcal{V}\to\infty}{\longrightarrow} \frac{1}{\mathcal{V}} \frac{\mathcal{V}}{(2\pi)^3} \int d\mathbf{k}\, f_\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}} = \int \frac{d\mathbf{k}}{(2\pi)^3}\, f_\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{A.6}$$

Now you see why we choose to put $1/\mathcal{V}$ in front of $\sum_\mathbf{k}$. We have

$$f(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^3}\, f_\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad f_\mathbf{k} = \int d\mathbf{r}\, f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{A.7}$$

and also

$$\int \frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{r}} = \delta(\mathbf{r}), \qquad \int d\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}} = (2\pi)^3\, \delta(\mathbf{k}). \tag{A.8}$$

Note that the dimensions are okay. Again it is easy to use these expression to verify that Fourier transforming twice brings you back to the starting point.

A.3 Time and frequency Fourier transforms

The time t and frequency ω transforms can be thought of as an extension of functions periodic with the finite period $\mathcal{T}$, to the case where this period tends to infinity. Thus t plays the role of $\mathbf{r}$ and ω that of $\mathbf{k}$, and in complete analogy with Eq. (A.7) – but with the opposite sign of i due to Eq. (A.1) – we have

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, f_\omega\, e^{-i\omega t}, \qquad f_\omega = \int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t}, \tag{A.9}$$

and also

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} = \delta(t), \qquad \int_{-\infty}^{\infty} dt\, e^{i\omega t} = 2\pi\, \delta(\omega). \tag{A.10}$$

Note again that the dimensions are okay.

A.4 Some useful rules

We can think of Eqs. (A.5), (A.8) and (A.10) as the Fourier transform of the constant function $f = 1$ to delta functions (and back):

$$1_\mathbf{r} \longleftrightarrow \mathcal{V}\, \delta_{\mathbf{k},0}, \qquad 1_\mathbf{k} \longleftrightarrow \delta(\mathbf{r}), \qquad \text{discrete } \mathbf{k}, \tag{A.11a}$$

$$1_\mathbf{r} \longleftrightarrow (2\pi)^3\, \delta(\mathbf{k}), \qquad 1_\mathbf{k} \longleftrightarrow \delta(\mathbf{r}), \qquad \text{continuous } \mathbf{k}, \tag{A.11b}$$

$$1_t \longleftrightarrow 2\pi\, \delta(\omega), \qquad 1_\omega \longleftrightarrow \delta(t), \qquad \text{continuous } \omega. \tag{A.11c}$$

Another useful rule is the rule for Fourier transforming convolution integrals. By direct application of the definitions and Eq. (A.8) we find

$$f(\mathbf{r}) = \int d\mathbf{s}\, h(\mathbf{r}-\mathbf{s})\, g(\mathbf{s}) = \int d\mathbf{s}\, \frac{1}{\mathcal{V}^2} \sum_{\mathbf{k},\mathbf{k}'} h_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{s})} g_{\mathbf{k}'} e^{i\mathbf{k}'\cdot\mathbf{s}} = \frac{1}{\mathcal{V}} \sum_{\mathbf{k}} h_{\mathbf{k}} g_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{A.12}$$

or in words: a convolution integral in **r**-space becomes a product in **k**-space.

$$\int d\mathbf{s}\, h(\mathbf{r}-\mathbf{s})\, g(\mathbf{s}) \;\longleftrightarrow\; h_{\mathbf{k}}\, g_{\mathbf{k}}. \tag{A.13}$$

A related rule, the invariance of inner products going from **r** to **k**, is derived in a similar way (and here given in three different versions):

$$\int d\mathbf{r}\, h(\mathbf{r})\, g^*(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^3}\, h_{\mathbf{k}} g^*_{\mathbf{k}}, \tag{A.14}$$

$$\int d\mathbf{r}\, h(\mathbf{r})\, g(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^3}\, h_{\mathbf{k}} g_{-\mathbf{k}}, \tag{A.15}$$

$$\int d\mathbf{r}\, h(\mathbf{r})\, g(-\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^3}\, h_{\mathbf{k}} g_{\mathbf{k}}. \tag{A.16}$$

Finally we mention the Fourier transformation of differential operators. For the gradient operator we have

$$\nabla_{\mathbf{r}} f(\mathbf{r}) = \nabla_{\mathbf{r}} \frac{1}{\mathcal{V}} \sum_{\mathbf{k}} f_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{\mathcal{V}} \sum_{\mathbf{k}} f_{\mathbf{k}}\, \nabla_{\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{\mathcal{V}} \sum_{\mathbf{k}} i\mathbf{k} f_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{A.17}$$

Similarly for ∇^2, $\nabla\times$, and ∂_t (remember the sign change of i in the latter):

$$\nabla_{\mathbf{r}} \;\longleftrightarrow\; i\mathbf{k}, \qquad \partial_t \;\longleftrightarrow\; -i\omega, \tag{A.18}$$

$$\nabla^2 \;\longleftrightarrow\; -\mathbf{k}^2, \qquad \nabla\times \;\longleftrightarrow\; i\mathbf{k}\times. \tag{A.19}$$

A.5 Translation invariant systems

We study a translation invariant system. Any physical observable $f(\mathbf{r},\mathbf{r}')$ of two spatial variables $\mathbf{r}$ and $\mathbf{r}'$ can only depend on the difference between the coordinates and not on the absolute position of any of them,

$$f(\mathbf{r},\mathbf{r}') = f(\mathbf{r}-\mathbf{r}'). \tag{A.20}$$

The consequences in **k**-space from this constraint are:

$$f(\mathbf{r},\mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\mathbf{k}'}{(2\pi)^3}\, f_{\mathbf{k},\mathbf{k}'}\, e^{i\mathbf{k}\cdot\mathbf{r}} e^{i\mathbf{k}'\cdot\mathbf{r}'} = \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\mathbf{k}'}{(2\pi)^3}\, f_{\mathbf{k},\mathbf{k}'}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} e^{i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{r}'}. \tag{A.21}$$

Since this has to be a function of $\mathbf{r}-\mathbf{r}'$, it is obvious from the factor $e^{i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{r}'}$ that any reference to the absolute value of $\mathbf{r}'$ only can vanish if $\mathbf{k}' = -\mathbf{k}$, and thus $f_{\mathbf{k},\mathbf{k}'} \propto \delta(\mathbf{k}+\mathbf{k}')$.

To find the proportionality constant, we can also find the Fourier transform of f by explicitly using that f only depends on the difference $\mathbf{r} - \mathbf{r}'$

$$f(\mathbf{r}, \mathbf{r}') = \tilde{f}(\mathbf{r} - \mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^3}\, \tilde{f}_{\mathbf{k}}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}, \tag{A.22}$$

and by comparing the two expressions Eqs. (A.21) and (A.22) we read off that

$$f_{\mathbf{k},\mathbf{k}'} = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}')\tilde{f}_{\mathbf{k}}, \tag{A.23}$$

or in short

$$f(\mathbf{r}, \mathbf{r}') \longleftrightarrow f_{\mathbf{k},-\mathbf{k}}, \qquad \text{translation} - \text{invariant systems.} \tag{A.24}$$

For the discrete case, we can go through the same arguments or use the formulae from above to get

$$f_{\mathbf{k},\mathbf{k}'} = \mathcal{V}\delta_{\mathbf{k},-\mathbf{k}'}\tilde{f}_{\mathbf{k}}. \tag{A.25}$$

This result is used several times in the main text when we consider correlation functions of the form

$$g(\mathbf{r}, \mathbf{r}') = \langle \mathcal{A}(\mathbf{r})\mathcal{B}(\mathbf{r}')\rangle, \tag{A.26}$$

where $\mathcal{A}$ and $\mathcal{B}$ are some operators. For a translation-invariant system we now that $g(\mathbf{r}, \mathbf{r}') = g(\mathbf{r} - \mathbf{r}')$, and by using the result in Eq. (A.25) for $\mathbf{k} = -\mathbf{k}'$ we get that

$$g(\mathbf{k}) = \frac{1}{\mathcal{V}}\langle \mathcal{A}(\mathbf{k})\mathcal{B}(-\mathbf{k})\rangle. \tag{A.27}$$