Special Relativity of Electric and Magnetic Fields

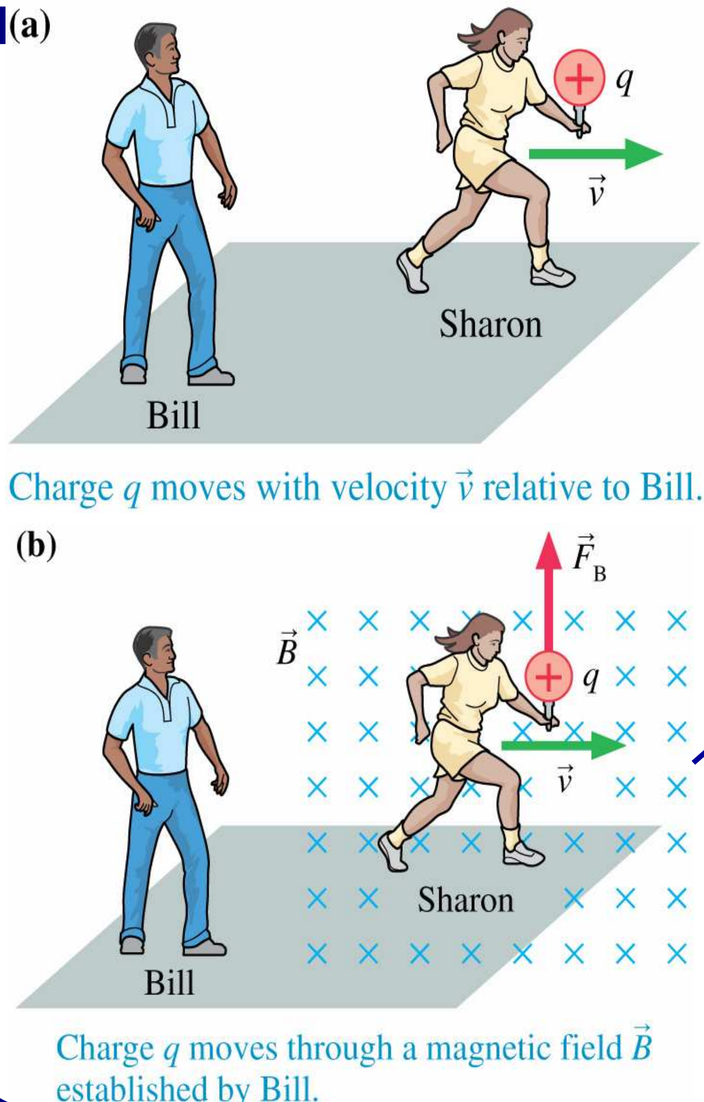
Branislav K. Nikolić

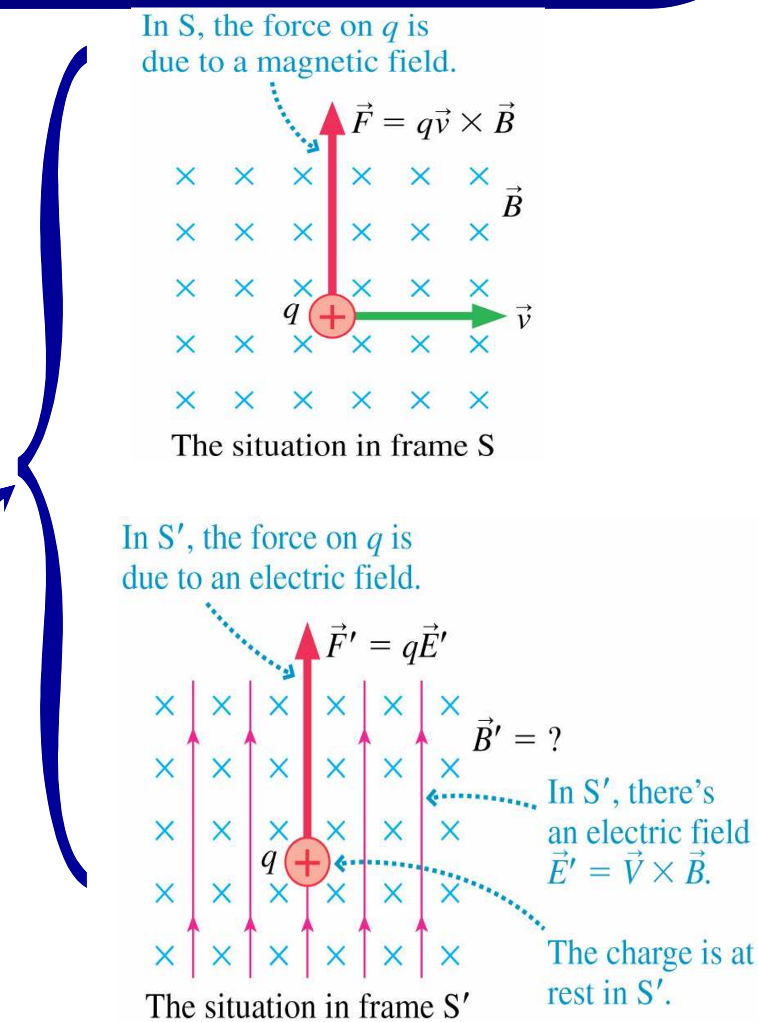
Department of Physics and Astronomy, University of Delaware, U.S.A.

PHYS 208 Honors: Fundamentals of Physics II

http://www.physics.udel.edu/~bnikolic/teaching/phys208/phys208.html

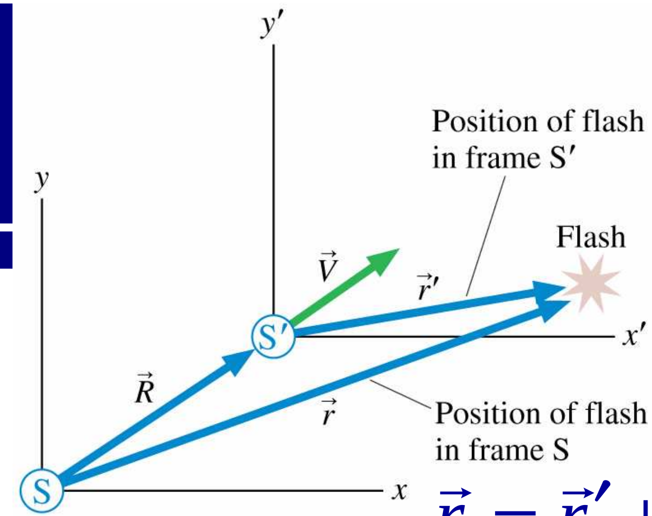
Electric and Magnetic Fields Depend on the Reference Frame



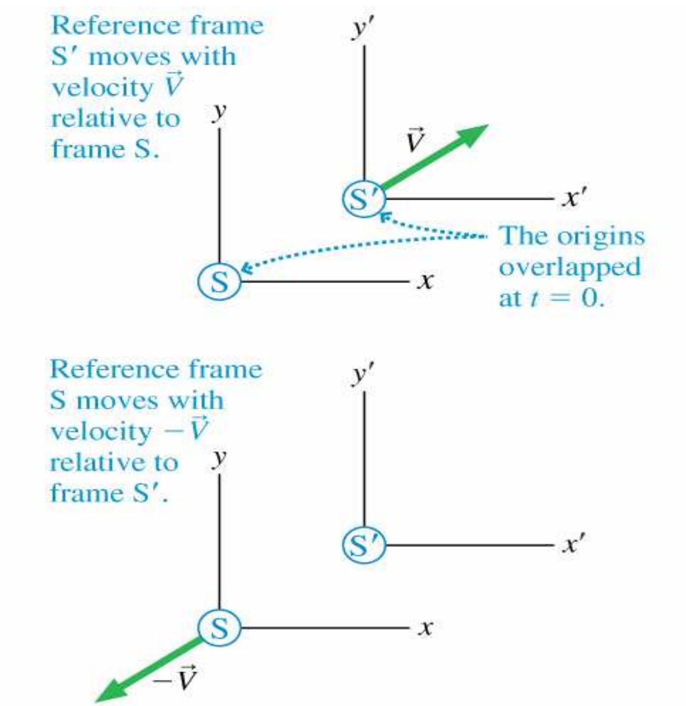


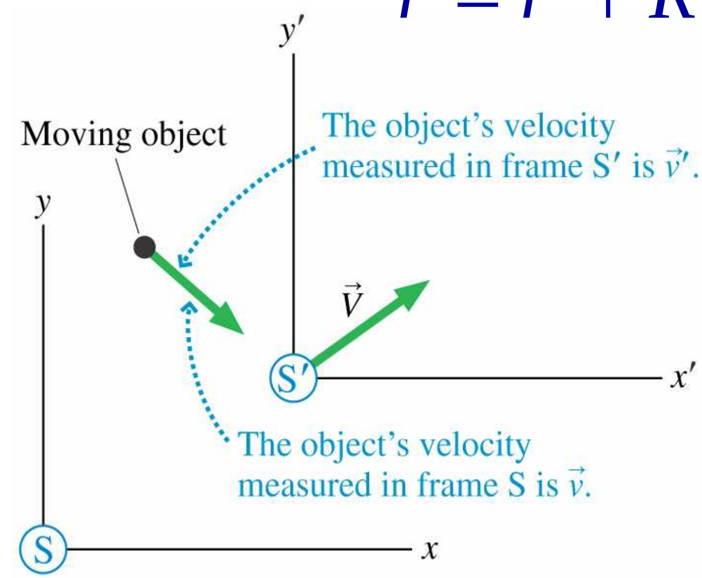
Galilean Relativity

$$\frac{d}{dt}\vec{v} = \frac{d}{dt}\vec{v}' \Leftrightarrow \vec{a} = \vec{a}' \Leftrightarrow \vec{F} = m\vec{a} = m\vec{a}' = \vec{F}'$$



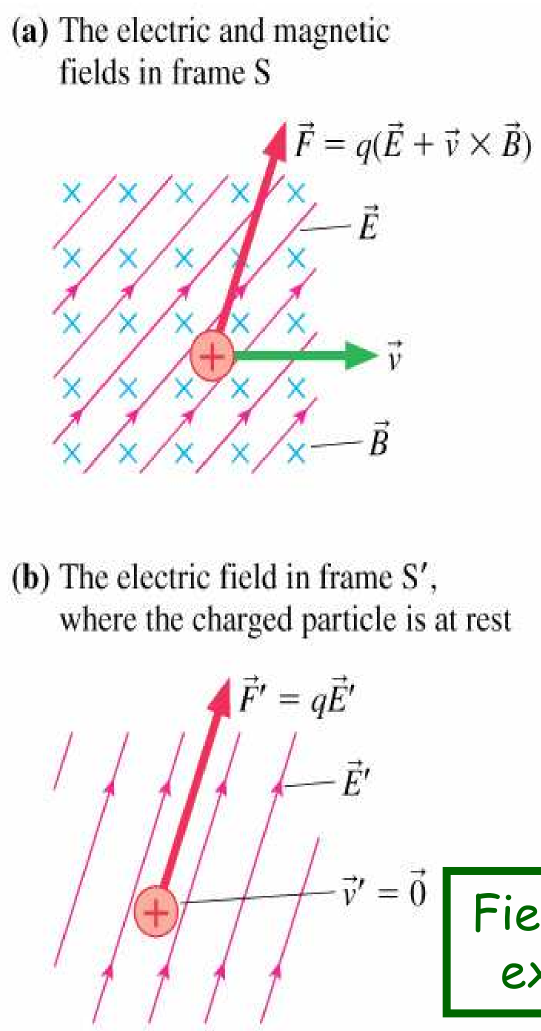
$$\vec{r} = \vec{r}' + \vec{R}$$





$$\frac{d}{dt}\vec{r} = \frac{d}{dt}\vec{r}' + \frac{d}{dt}\vec{R} \Leftrightarrow \vec{v} = \vec{v}' + \vec{V}$$

"Galilean Transformations" of E and B Fields: Weak Relativistic Approximation ($v \ll c$)



$$v/c \ll 1 \Rightarrow \vec{F} = \vec{F}'$$

$$\Downarrow$$

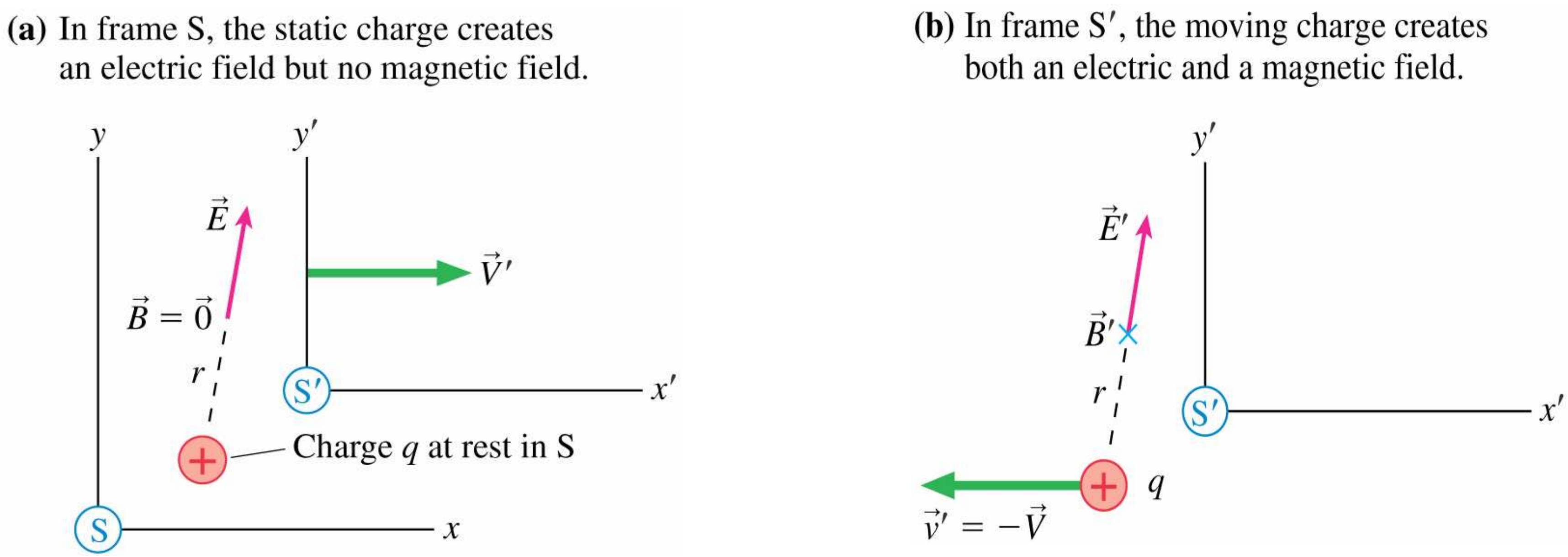
$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$$

- To get transformation formulas for magnetic field one has to use full special relativity derivation and then take its limit for $v \ll c$

$$\vec{B}' = \vec{B} - \frac{1}{c^2}\vec{v} \times \vec{E}$$

Fields are measured at the same point in space by experimenters at rest in each reference frame

Bio-Savart Law as Coulomb Law Transformed Into Moving Reference Frame

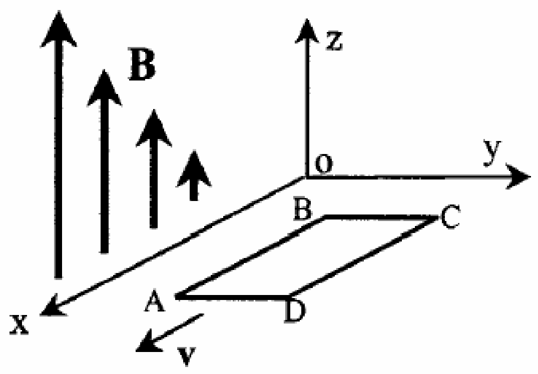


$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}, \quad \vec{B} = 0$$

$$\vec{E}' = \vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

Biot-Savar: $\vec{B}' = \frac{\mu_0}{4\pi}\frac{q}{r^2}\vec{v}\times\vec{r} = -\frac{\mu_0}{4\pi}\frac{q}{r^2}\vec{V}\times\vec{r} = -\frac{1}{c^2}\vec{V}\times\vec{E}$

Faraday's Law of EM Induction Revisited



A rectangular conducting loop $ABCD$ (sides b and d) moves along the x axis in a plane through a magnetic field with a linearly increasing intensity $\mathbf{B}$ $(0, 0, B_0 x)$.

Motional EMF detected by an inertial observer **may appear** as a curly EMF to another observer

LAB Frame:

$$x_1 = vt,\ x_2 = b + vt$$

$$\vec{B}_{L,1} = (0, 0, B_0 vt),\ \vec{B}_{L,2} = (0, 0, B_0 b + B_0 vt)$$

$$\varepsilon_{motional} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{s} = -vB_0 A$$

LOOP Frame:

$$\text{no motion, } B_F^z = B_0 (x_F + vt)$$

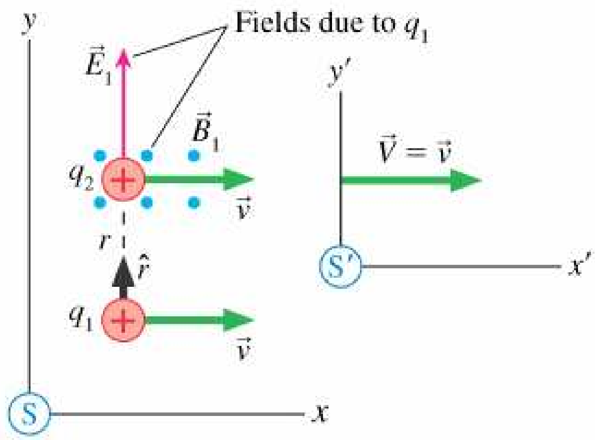
$$\frac{\partial \vec{B}_F}{\partial t} = (0, 0, B_0 v)$$

$$\varepsilon_{curly} = -\iint_A \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} = -vB_0 A$$

$$\varepsilon_{curly} = \oint_{ABCD} \vec{E}_F \cdot d\vec{s} = \oint_{ABCD} (\vec{E}_L + \vec{v} \times \vec{B}) \cdot d\vec{s}$$

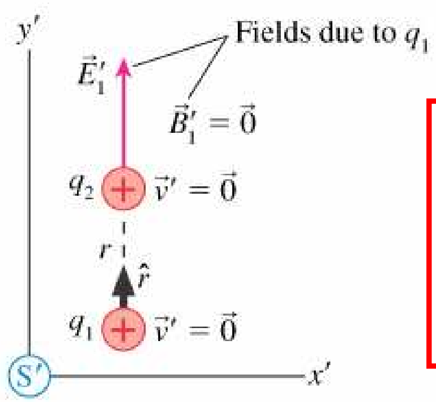
$$\text{for } v \ll c$$

Almost Special Relativity



Fields seen in frame S

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{j},\ \vec{B}_1 = \frac{\mu_0}{4\pi}\frac{q_1 v}{r^2}\hat{k}$$



Fields seen in frame S′

$$\vec{E}'_1 = \vec{E}_1 + \vec{V}\times\vec{B}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{j} + v\hat{i}\times\frac{\mu_0}{4\pi}\frac{q_1 v}{r^2}\hat{k} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\left(1 - \frac{v^2}{c^2}\right)\hat{j}$$

?

$$\vec{B}'_1 = \vec{B}_1 - \frac{1}{c^2}\vec{V}\times\vec{E}_1 = \frac{\mu_0}{4\pi}\frac{q_1 v}{r^2}\hat{k} - \frac{1}{c^2}\left(v\hat{i}\times\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{j}\right) = \frac{\mu_0}{4\pi}\frac{q_1 v}{r^2}\left(1 - \frac{1}{\varepsilon_0\mu_0 c^2}\right)\hat{k} \equiv 0$$

Lorentz-Einstein Transformations of Electric and Magnetic Fields

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}$$

$$\vec{E}'_{\perp} = \frac{\vec{E}_{\perp} + \vec{V} \times \vec{B}}{\sqrt{1 - v^2/c^2}}$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

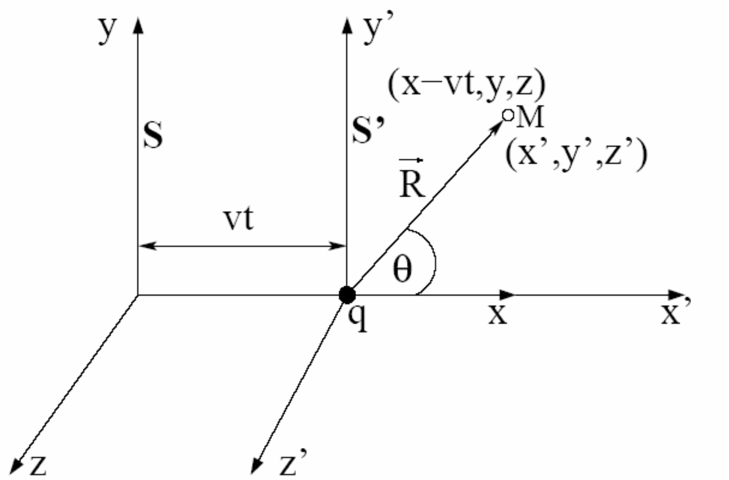
$$\vec{B}'_{\perp} = \frac{\vec{B}_{\perp} - \vec{V} \times \vec{E}/c^2}{\sqrt{1 - v^2/c^2}}$$

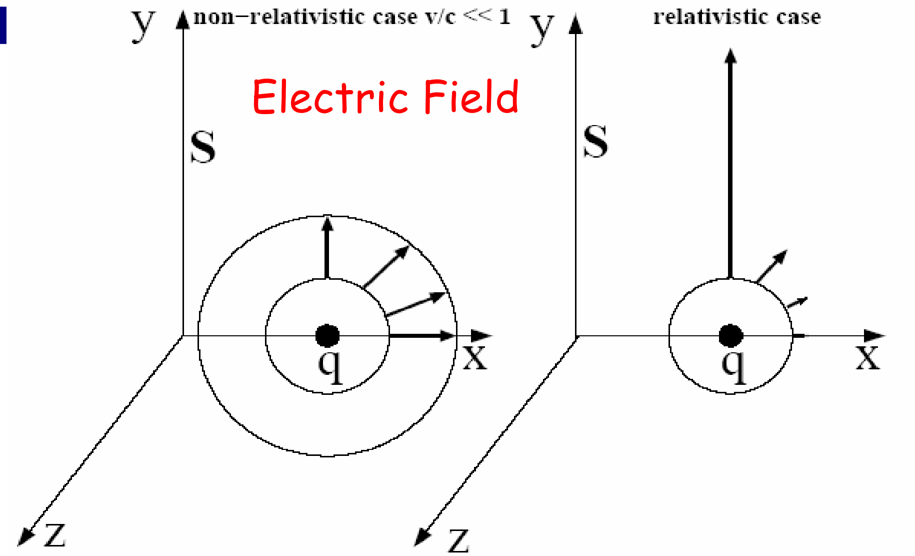
Electric and Magnetic Fields are different facets of a single electromagnetic field whose particular manifestation (and division into its E and B components) depends largely on the chosen reference frame!

- Simple Corollaries:

1. $\vec{E} \neq 0, \vec{B} = 0 \Rightarrow \vec{B}' = \vec{B}'_{\parallel} + \vec{B}'_{\perp} = -\vec{V} \times \vec{E}'/c^2$
2. $\vec{B} \neq 0, \vec{E} = 0 \Rightarrow \vec{E}' = \vec{V} \times \vec{B}'$
3. EM Field Invariants: $\vec{E} \cdot \vec{B} = const.; E^2 - c^2 B^2 = const.$

Electromagnetic Field of Freely Moving Relativistic Charge





MOVING (with charge) S′-Frame:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{R}'}{R'^3}$$

$$\vec{B} = 0$$

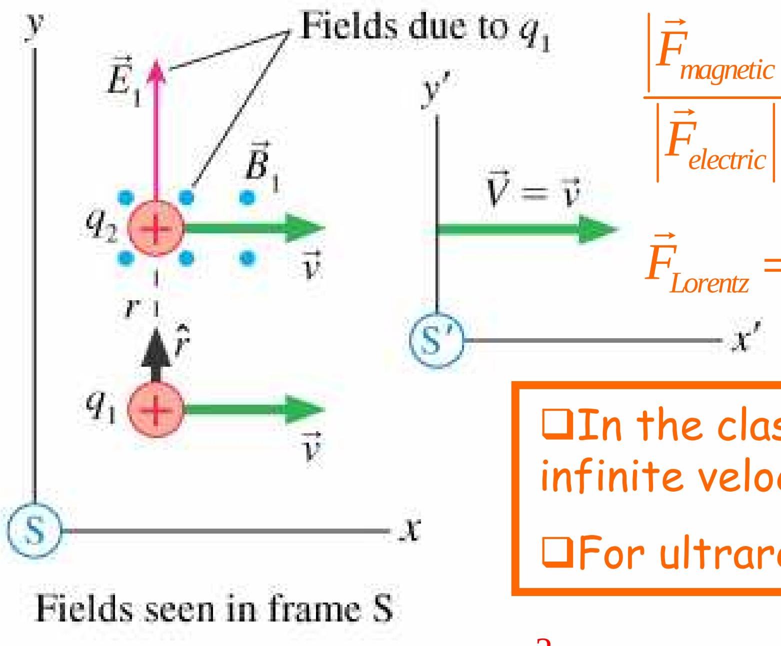
LAB S-Frame:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1 - v^2/c^2}{(1 - v^2\sin^2\theta/c^2)^{3/2}}\frac{\vec{R}}{R^3}$$

$$x - vt = R\cos\theta, \sqrt{y^2 + z^2} = R\sin\theta$$

$$\vec{B} = \frac{1}{c^2}\vec{v}\times\vec{E}$$

Electromagnetic Force Between Two Moving Charges



$$\frac{\left|\vec{F}_{magnetic}\right|}{\left|\vec{F}_{electric}\right|} = \varepsilon_0 \mu_0 v^2 = \frac{v^2}{c^2}$$

$$\vec{F}_{Lorentz} = \vec{F}_{electric} + \vec{F}_{magnetic} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\sqrt{1-v^2/c^2}\,\hat{j} = \vec{F}'_{Lorentz}\sqrt{1-v^2/c^2}$$

- In the classical Galileo-Newton world signals propagate at infinite velocity ($c \to \infty$) and **magnetism is absent**!
- For ultrarelativistic particles $v \to c \Rightarrow F_{magnetic} \to F_{electric}$

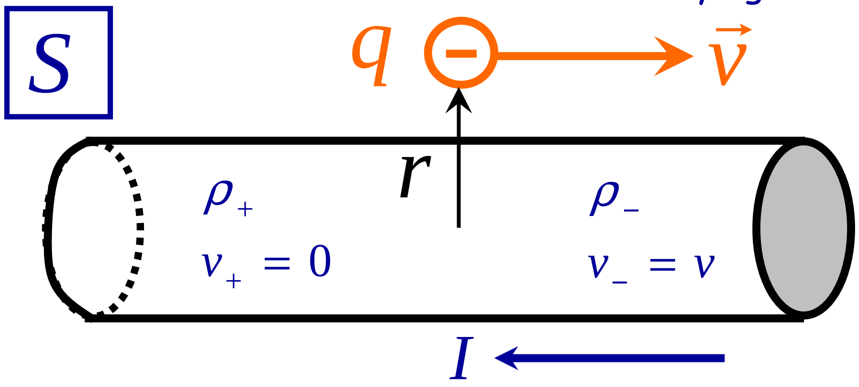
$$\vec{F}_{electric} = q\vec{E}_1 = \frac{q^2}{4\pi\varepsilon_0}\frac{1-v^2/c^2}{(1-v^2\sin^2 90^\circ/c^2)^{3/2} r^2}\hat{j} = \frac{q^2}{4\pi\varepsilon_0}\frac{1}{r^2\sqrt{1-v^2/c^2}}\hat{j}$$

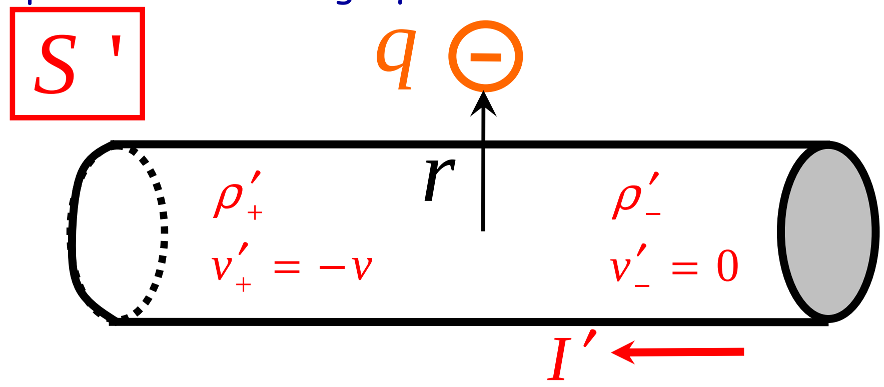
$$\vec{F}_{magnetic} = q\vec{v}\times\vec{B}_1 = qv\frac{1}{c^2}vE_1(-\hat{j}) = -\frac{\mu_0}{4\pi}\frac{q^2v^2}{r^2\sqrt{1-v^2/c^2}}\hat{j}$$

Transformation Laws for Charge and Current Densities

$$\rho = \frac{\rho_{proper}}{\sqrt{1 - v^2/c^2}}, \quad \vec{J} = \rho\vec{v} = \frac{\rho_{proper} v}{\sqrt{1 - v^2/c^2}}$$

- Interaction of current a current carrying wire and a particle with charge q in two intertial frames:





$$\vec{F}_{magnetic} = q\vec{v} \times \vec{B}$$

$$F = \frac{1}{4\pi\varepsilon_0 c^2}\frac{2Iqv}{r}$$

$$I = \rho_- vA \Rightarrow F = \frac{q}{2\pi\varepsilon_0}\frac{\rho_- A}{r}\frac{v^2}{c^2}$$

Force Transformation Law:

$$\frac{\Delta p'_y}{\Delta p_y} = \frac{F'\Delta t'}{F\Delta t}, \Delta t = \frac{\Delta t'}{\sqrt{1-v^2/c^2}} \Rightarrow F' = \frac{F}{\sqrt{1-v^2/c^2}}$$

$$\vec{F}'_{magnetic} = 0, \vec{F}'_{electric} \neq 0$$

$$\underbrace{Q = Q', \rho LA = \rho' L' A, L' = L\sqrt{1-v^2/c^2}}_{\Downarrow}$$

$$\rho'_+ = \frac{\rho_+}{\sqrt{1-v^2/c^2}}, \quad \rho'_- = \rho_-\sqrt{1-v^2/c^2}, \quad \rho_- = -\rho_+$$

$$E' = \frac{(\rho'_+ + \rho'_-)A}{2\pi\varepsilon_0 r} = \frac{\rho_+ A v^2/c^2}{2\pi\varepsilon_0 r\sqrt{1-v^2/c^2}}$$

$$F' = qE' = \frac{q}{2\pi\varepsilon_0}\frac{\rho_+ A}{r}\frac{v^2/c^2}{\sqrt{1-v^2/c^2}} = \frac{F}{\sqrt{1-v^2/c^2}}$$