PHYS 812: Assignment 6

1. Demonstrate that the Schrödinger equation is gauge invariant. Start from the Hamiltonian $H$ for a particle in the electromagnetic field defined by some arbitrary potentials $A$ and $\phi$.

   (a) Write down $H_A$, the Hamiltonian obtained by gauge transforming the potentials $A$ and $\phi$ into: $A' = A - \nabla \Lambda$ and $\phi' = \phi + (1/c) \partial \Lambda / \partial t$, where $\Lambda(r, t)$ is an arbitrary real function.

   (b) Show that if $\psi(r, t)$ is a solution to Schrödinger’s equation with the Hamiltonian $H$, then $\psi_A = e^{-i\eta \Lambda / \hbar} \psi$ is the corresponding solution after the gauge transformation, i.e., it contains the same physical information.

2. Consider the photoelectric effect for the hydrogen atom and hydrogen-like ions initially in the ground state.

   (a) For hydrogen atom, estimate the photoelectric cross-section $\sigma$ when the ejected electron has a kinetic energy of 10 Ry. Compare it to the atom’s geometric cross-section of about $\pi a_0^2$.

   (b) For an ion of charge $Z$, show that $\sigma$ is proportional to $Z^5$ in the limit of $p_f a_0 / (Z \hbar) \gg 1$.

3. Consider the classical Hamiltonian of the electromagnetic field

   $$\mathcal{H} = \frac{1}{8\pi} \int \left( |E|^2 + |B|^2 \right) d^3 r$$

   where $E$ and $B$ are the electric and magnetic field, respectively. For the field with no sources and in gauge $\phi = 0$, $\mathcal{H}$ can be written as

   $$\mathcal{H} = \frac{1}{8\pi c^2} \int \left( |\dot{A}|^2 + c^2 |\nabla \times A|^2 \right) d^3 r$$

   where $A$ is the vector potential and $c$ is the speed of light. Show that if the vector potential is expressed in the form

   $$A(r, t) = \sum_{\lambda=1}^{2} \int \sqrt{\frac{e^2}{4\pi^2 \omega}} \left[ a_{\lambda}(k) \hat{e}_\lambda(k) e^{i(k \cdot r - \omega t)} + a_{\lambda}^*(k) \hat{e}_\lambda(k) e^{-i(k \cdot r - \omega t)} \right] d^3 k$$

   where $k$ is the wave vector, $\omega$ is the angular frequency of the field, $\omega = kc$, and $\hat{e}_\lambda(k)$ are unit vectors ($a_3 = 0$ by assumption), the Hamiltonian becomes

   $$\mathcal{H} = \sum_{\lambda=1}^{2} \int \omega [a_{\lambda}^*(k) a_{\lambda}(k)] d^3 k.$$
which is non-Hermitian. Thus, one has to first symmetrize $\mathcal{H}$ and then make the substitution.

The resulting Hermitian Hamiltonian can be reduced to the form

$$ H = \sum_{\lambda=1}^{2} \int \hbar \omega \left[ \alpha^{\dagger}_\lambda(k) \alpha_\lambda(k) + \frac{1}{2} \right] d^3k. $$

by using the commutation relation for the creation and annihilation operators: $[a_\lambda(k), a^{\dagger}_{\lambda'}(k')] = \delta_{\lambda,\lambda'} \delta^{(3)}(k - k')$. However, this commutation relation cannot be used in the final expression for $\mathcal{H}$ (from the previous problem) since it has been integrated over $k'$.

(a) One may think that the problem can be solved by starting from an earlier equation in the derivation of $\mathcal{H}$, before the integration over $k'$. Try this approach and show that it leads to an expression containing the square of the Dirac delta function, which is not a well-defined quantity.

(b) To solve this paradox, use a discrete approach, i.e., represent the vector potential in the form:

$$ A(r) = c \sqrt{\frac{2\pi}{V}} \sum_{\lambda=1}^{2} \sum_{k} \frac{1}{\omega} \left[ a_\lambda(k) \hat{e}_\lambda(k) e^{ik \cdot r} + a^{\dagger}_\lambda(k) \hat{e}_\lambda(k) e^{-ik \cdot r} \right] $$

with each component of $k$ going over a set of discrete values $k_i = \frac{2\pi n_i}{L}, \; i = x, y, z, n_i = 0, \pm 1, \pm 2, \ldots$.

5. Let us denote the operator which creates [annihilates] a photon of momentum $\hbar k$ and polarization $\lambda$ ($\lambda = 1$ or 2) by $a^{\dagger}(k\lambda)$ [$a(k\lambda)$]. Using the commutation rule for these operators $[a(k\lambda), a^{\dagger}(k'\lambda')] = \delta_{\lambda,\lambda'} \delta^{(3)}(k - k')$

(all the other commutators are zero)

(a) Show that the states created are eigenstates of the field Hamiltonian

$$ H = \sum_{\lambda} \int \left[ a^{\dagger}(k\lambda) a(k\lambda) + \frac{1}{2} \right] \hbar \omega \; d^3k $$

where $\omega = kc$.

(b) Find normalization of these states.

(c) Show that photons are bosons.