PHYS 812: Assignment 1

1. Show that time translational invariance implies conservation of energy.
   
   (a) Define space and time translations and find their generators.
   
   (b) Define first space translational invariance. Explain why an analogous definition cannot
   be used for time translational invariance. Then give a definition of time translational
   invariance.
   
   (c) Find the condition for the Hamiltonian resulting from point (b). It is sufficient to
   consider an infinitesimal time translation.
   
   (d) Use Ehrenfest’s theorem to show that energy is conserved if this condition is fulfilled.
   
   (e) Apply Ehrenfest’s theorem to the generator of space translation. Where do you use
   space translation invariance to get the conservation of momentum theorem? Then
   apply Ehrenfest’s theorem to the generator of time translation. How is the use of the
   invariance different from the space translation case?

2. The WKB quantization condition for the potential with finite derivatives at both turning
   points,
   
   \[ \int_{x_a}^{x_b} p(x)dx = (n + \frac{1}{2})\pi\hbar, \quad n = 0, 1, 2, \ldots, \quad (1) \]
   
   does not strictly apply to the hydrogen atom (for several reasons listed below). A more
   rigorous condition is due to Maslov: \( \int_{x_a}^{x_b} p(x)dx = (n + l + 1 - \sqrt{l(l+1)})\pi\hbar \). Derive this
   condition for the case of \( l = 0 \).

   (a) Start from relating the WKB wave function to Schrödinger’s radial wave function for
   spherically symmetric problems: \( \psi_{nlm}(r) = R_{nl}(r)Y_{lm}(\theta, \phi) \). Should one use \( R(r) \) or
   \( u(r) = rR(r) \) ?
   
   (b) Now explain why neither the condition (1) nor the condition \( \int_{x_a}^{x_b} p(x)dx = (n + \frac{3}{2})\pi\hbar \)
   derived for the potentials with \( V(r) = \infty \) for \( r \leq 0 \) applies to the hydrogen atom with
   \( l = 0 \).
   
   (c) One may think that the condition (1) should apply to the case \( l > 0 \). Explain why this
   is not the case either.
   
   (d) To derive Maslov’s condition for \( l = 0 \), consider the asymptotic behavior of
   Schrödinger’s radial equation for \( r \to 0 \) in analogy with the derivation of the condition (1).
   \( \text{Hints: Transform this equation by substitutions } \rho = (2me^2/\hbar^2)r = z^2/4 \)
   and \( u(\rho) = zv(z) \) into the equation \( x^2y'' + xy' + [x^2 - k^2]y = 0 \) for the Bessel functions
   \( J_k(x) \). Then use the asymptotic form of these functions.
   
   (e) Use the formula obtained in (d) to compute \( l = 0 \) energy levels of hydrogen atom. \( \text{Hint:} \)
   \[ \int_0^1 \sqrt{\frac{1}{x} - 1}dx = \frac{\pi}{2}. \]
3. A charged harmonic oscillator of mass $m$ and charge $q$, described by the unperturbed Hamiltonian $H_0 = P^2/(2m) + \frac{1}{2}m\omega^2$, is placed in a constant external electric field, i.e., the perturbation operator is $V = -qE X$. As shown in class, this problem can be solved exactly and the exact energy of the $n$th state of the system is $E_n = (n + \frac{1}{2})\hbar\omega - q^2E^2/(2m\omega^2)$, with the second term identical to that obtained in the second order perturbation theory. The exact wave function was shown to be $|n\rangle = \exp(-iqE\sqrt{\hbar/(2m\omega)}|n(0)\rangle)$, where $|n(0)\rangle$ is the solution of the unperturbed problem. You may find it convenient to express $X$ and $P$ in terms of lowering and raising operators: $X = \sqrt{\hbar/(2m\omega)}(a + a^\dagger)$ and $P = \frac{i}{\sqrt{m\omega/(2\hbar)}}(a - a^\dagger)$, with $a|n(0)\rangle = \sqrt{n}|(n-1)(0)\rangle$ and $a^\dagger|n(0)\rangle = \sqrt{n+1}|(n+1)(0)\rangle$. Do not prove any of the relations discussed above.

(a) Show that the energy relation implies that all the energy corrections $E^{(k)}_n, k \geq 3$, must be zero, i.e., the other option that these corrections are nonzero but add up to zero is not possible.

(b) On the other hand, show by expanding the exact function in power series that all the wave function corrections, $n^{(k)}$, are nonzero.

(c) The only conclusion from (a) and (b) is that all the expressions for $E^{(k)}_n, k \geq 3$, have the value of zero. Show this explicitly for $E^{(3)}_n$ using the proper order wave function correction obtained from the expansion of point (b) and the perturbation theory expression for $E^{(3)}_n$. Note that the wave function corrections obtained in such a way to not satisfy, in general, the intermediate normalization condition, so you have to derive a formula for $E^{(3)}_n$ without assuming any normalization.

(d) So we are now completely convinced that the odd-order energy corrections are zero. What about even-order ones? $E^{(2)}_n$ is nonzero. Find the value of $E^{(4)}_n$ by an explicit calculation.