1. Show that the time evolution of an initial state $\psi(x,0)$ of a particle of mass $m$ in a one-dimensional potential $V(x)$ is the same in Schrödinger’s and Path Integral (PI) formulations of Quantum Mechanics. Hint: The PI propagator can be written as:

$$U(x_N, t; x_0, 0) = \lim_{N \to \infty} \left( \frac{m}{2\pi\hbar\epsilon} \right)^{N/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \times \exp \left[ \frac{i}{\hbar} \left( \sum_{k=0}^{N-1} \frac{m}{2} \frac{(x_{k+1} - x_k)^2}{\epsilon} - \epsilon V(x_k) \right) \right] dx_1 \cdots dx_{N-1},$$

where $x_k = x(t_k)$ and $t_{k+1} - t_k = \epsilon$. It is sufficient to consider only the propagation over infinitesimal time, in which case $N = 1$ and no integration over intermediate $x$ is needed.

2. A point in $xy$ plane is undergoing the following four transformation: (a) translation by $\epsilon^x \hat{x}$; (b) counterclockwise rotation by angle $\epsilon_z$ around the $z$ axis; (c) translation by $-\epsilon^x \hat{x}$; (d) counterclockwise rotation by angle $-\epsilon_z$ around the $z$ axis.

(a) Find the coordinates of the point $(x, y)^T$ after this set of transformations keeping only the terms of order up to $\epsilon_x \epsilon_z^2$. Is there a single transformation that is equivalent to the set of four transformations defined above?

(b) Find the quantum mechanical operators corresponding to all the discussed transformations and expand these operators up to $\epsilon_x \epsilon_z^2$.

(c) By equating coefficients of $\epsilon_x \epsilon_z^2$, deduce the constraint

$$-2L_z P_x L_z + P_x L_z^2 + L_z P_x = \hbar^2 P_x$$

resulting from these transformations.

(d) Use the identity

$$-2\Lambda \Omega \Lambda + \Omega \Lambda^2 + \Lambda^2 \Omega = [\Lambda, [\Lambda, \Omega]]$$

to verify that the constraint from point (c) is satisfied given the commutation relations between $P_x$, $P_y$, and $L_z$.

3. Consider a rotation defined in terms of Euler angles $\alpha$, $\beta$, $\gamma$, with the corresponding rotation operator

$$U[R_{\alpha\beta\gamma}] = e^{-i\alpha J_z/\hbar} e^{-i\beta J_y/\hbar} e^{-i\gamma J_z/\hbar},$$

where $J_i$ are components of the angular momentum operator.
(a) Construct the \( \mathbb{J}^{(1)} \) matrices, i.e., the matrices of \( J_i \) in the space \( \mathbb{V}^{(1)} \) of \( J^2 \) and \( J_z \) eigenkets \( |jm\rangle \) with \( j = 1 \), and use the relation

\[
\mathbb{D}^{(1)}[R_{\theta, \varphi, \chi}] = (\cos \theta_x - 1) \left( \mathbb{J}^{(1)} \right)^2 - i \sin \theta_x \frac{\mathbb{J}^{(1)}_{x} \mathbb{J}^{(1)}_{y}}{\hbar} + \mathbb{I}^{(1)},
\]

where \( \mathbb{D}^{(1)}[R_{\theta, \varphi, \chi}] \) is the matrix of the operator \( U[R_{\theta, \varphi, \chi}] \) in \( \mathbb{V}^{(1)} \), valid also for the rotation around the other axes, to find the matrix \( \mathbb{D}^{(1)}[R_{\alpha, \beta, \gamma}] \) (matrix of the operator \( U[R_{\alpha, \beta, \gamma}] \) in \( \mathbb{V}^{(1)} \)). You may leave this matrix in the form of a product of three matrices.

(b) Use this matrix to find the result of rotating the ket \( |jm\rangle = |11\rangle \) by angles \( \alpha, \beta, \gamma \).

(c) Find the expectation value of \( \langle J \rangle \) in the rotated state.

4. Consider a state \( |\psi\rangle \) represented by a spinor

\[
|\psi\rangle(r) = R(r) \left( \frac{Y^0_0(\theta, \phi)}{\sqrt{3}} + \frac{1}{\sqrt{3}} Y^0_1(\theta, \phi) \right).
\]

Starting from the standard definition for the probability of measuring the eigenvalue \( m \hbar \) of the operator \( L_z \)

\[
P(m \hbar) = \frac{\langle \psi | P_m | \psi \rangle}{\langle \psi | \psi \rangle},
\]

where \( P_m \) projects on all eigenstates of \( L_z \) with eigenvalue \( m \hbar \), find the possible outcomes of the measurement of \( L_z \) on this state and their probabilities.

5. (a) Write down the additional postulates of quantum mechanics resulting from consideration of spin of particles.

(b) Denote by \( |\hat{n} \pm\rangle \) the eigenkets of the operator that is the projection of the spin-1/2 operator \( \mathbf{S} \) on the direction of \( \hat{n} \). Find the equation determining the parameters of the expansion of these eigenkets in terms of eigenkets \( |\pm\rangle \) of \( S_z \), i.e., \( |\hat{n} \pm\rangle = \alpha_\pm |+\rangle + \beta_\pm |-\rangle \) in terms of spherical angles of \( \hat{n} \). Do not solve these equations.

(c) Using quantum mechanics, derive the interaction operator for motion of a particle of mass \( m \) in a weak uniform magnetic field along the \( \hat{z} \) axis and extend this formula to spin angular momentum using spin postulates.

6. Find the Clebsch-Gordan coefficients of \( 1 \otimes 1 = 2 \oplus 1 \oplus 0 \).