1. Consider a one-dimensional system.

(a) Find the action of the parity operator \( \Pi \), defined by

\[
| - x \rangle = \Pi | x \rangle
\]

on an arbitrary ket in the \( x \)-representation, i.e., find \( < x | \Pi \psi \rangle = \Pi \psi(x) \).

(b) Find \( | p \rangle \), where \( | p \rangle \) is an eigenket of the momentum operator.

(c) Prove that \( \Pi \) is Hermitean and unitary.

(d) Find the eigenvalues of this operator.

(e) Prove that the eigenfunctions of \( \Pi \) corresponding to different eigenvalues are orthogonal.

(f) Consider particle in a box. Under what circumstances does the Hamiltonian \( H \) commute with \( \Pi \)? What can you say about the eigenfunctions of \( H \) in this case?

2. Consider the time-independent Schrödinger equation for a particle of mass \( m \)

\[
-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x) \psi(x) = E \psi(x).
\]

The potential energy \( V(x) \) is defined by

\[
V(x) = \begin{cases} 
0 & \text{for } x < 0 \\
V_0 & \text{for } x > 0
\end{cases}
\]

where \( V_0 \) is a positive constant. The incident particle comes from \( x = -\infty \).

(a) Determine the most general form of the physically possible solutions with energy \( E > V_0 \) separately in each interval of \( x \).

(b) Identify components of the wave function representing particles moving to the left and to the right and then simplify the solutions based on information given above.

(c) Now find the solution valid in both regions.

(d) Determine the currents for all components of the wave function and used them to define the transmission \( T \) and reflection \( R \) coefficients.

(e) Calculate \( T \) and \( R \) and check if their sum is the expected value.
3. Consider the time-independent Hamiltonian $H$ for a simple harmonic oscillator:

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2,$$

where $m$ is the mass of the particle and $\omega$ is a constant.

(a) Rewrite the Hamiltonian in terms of the variable $\xi = \sqrt{m\omega/\hbar}x$.

(b) Now “factorize” the Hamiltonian such that $H/(\hbar\omega)$ is a product of two operators plus a constant. Denote the operators by $a$ and $a^\dagger$.

(c) Evaluate the commutators of this pair of operators with each other and with the Hamiltonian.

(d) Show that the lowest eigenvalue of the Hamiltonian is nonnegative.

(e) Use this fact and the commutation relations to find the lowest eigenvalue.