1. Consider the operator $J$ defined only by the commutation relations $J \times J = i\hbar J$. Using the basic properties of this operator proved in class, show that:

(i) $\langle J_x \rangle = \langle J_y \rangle = 0$ in a state $|jm\rangle$ that is an eigenstate of $J^2$ and $J_z$. 

(ii) Show that in these states $\langle J_x^2 \rangle = \langle J_y^2 \rangle = \frac{\hbar^2}{2} [j(j+1) - m^2]$ (use symmetry arguments to relate $\langle J_x^2 \rangle$ to $\langle J_y^2 \rangle$).

(iii) Check that $\Delta J_x \Delta J_y$ from part (ii) satisfies the inequality imposed by the uncertainty principle [Eq. (9.2.9)].

(iv) Show that the uncertainty bound is saturated in the state $|j, \pm j\rangle$.

2. Denote by $J_i^{(j)}$ the $j$th block of the matrix of the operator $J_i$, $i = x, y, z$, in the eigenbasis $|jm\rangle$ of the angular momentum operators $J^2$ and $J_z$.

(i) Argue that the eigenvalues of $J_x^{(j)}$ and $J_y^{(j)}$ are the same as those of $J_z^{(j)}$, namely, $\pm j$, $j(j-1), \ldots, \mp j$. Generalize the result to $J_\theta^{(j)} \equiv \vec{\theta} \cdot \vec{J}^{(j)}$.

(ii) Show that $\left[ J_\theta^{(j)} - j\hbar \right] \left[ J_\theta^{(j)} - (j-1)\hbar \right] \cdots \left[ J_\theta^{(j)} + j\hbar \right] = 0$.

Hint: In the case $J_\theta^{(j)} = J_z^{(j)}$, what happens when $K_\theta^{(j)}$ acts on a column vector $e_{jm}$ of length $j$ with elements $(e_{jm})_i = \delta_{im}$? What about an arbitrary superposition of such vectors?

(iii) Show that (ii) implies that $\left( J_\theta^{(j)} \right)^{2j+1}$ is a linear combination of $\left( J_\theta^{(j)} \right)^0, \left( J_\theta^{(j)} \right)^1, \ldots, \left( J_\theta^{(j)} \right)^{2j}$. Argue that the same goes for $\left( J_\theta^{(j)} \right)^{2j+k}$, $k = 2, 3, \ldots$.

(iv) Show that (iii) implies that the infinite sum in the definition of the matrices $D^{(j)}[R_\theta]$ of the rotation operator $U[R_\theta]$

$$D^{(j)}[R_\theta] = \exp(-i\theta \cdot \vec{J}^{(j)}/\hbar) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{i}{\hbar} \right)^n \left( \theta \cdot \vec{J}^{(j)} \right)^n$$

can be reduced to a finite sum.

3. Consider the rotation by an angle $\theta$ around vector $\hat{\theta}$. Let $U[R_\theta]$ be the operator corresponding to this rotation and acting on vectors in the Hilbert space. Denote the $j$th block of the matrix of this operator in the eigenbasis $|jm\rangle$ of the angular momentum operators $J^2$ and $J_z$ by $D^{(j)}[R_\theta]$. Assume that the relation between the matrix $D^{(j)}$ and the respective matrices $J_i^{(j)}$, $i = x, y, z$, of the components of the angular momentum operator is given by

$$D^{(j)}[R_\theta] = \exp(-i\theta \cdot \vec{J}^{(j)}/\hbar)$$

where $\theta \cdot \vec{J}^{(j)} = \theta_x J_x^{(j)} + \theta_y J_y^{(j)} + \theta_z J_z^{(j)}$. Show that
where $I^{(j)}$ is the unit matrix of dimension $2j + 1$.

(ii)

$$D^{(1)}[R_{\theta, \hat{x}}] = \begin{pmatrix} \cos(\theta_x - 1) \left( \frac{J^{(1)}_x}{\hbar} \right)^2 - i \sin(\theta_x) \frac{J^{(1)}_x}{\hbar} + \mathbb{1}^{(1)} \end{pmatrix}$$

Hint: Use the following relation for the matrices $J^{(j)}_\theta \equiv \hat{\theta} \cdot J^{(j)}$

$$[J^{(j)}_\theta - j\hbar][J^{(j)}_\theta - (j - 1)\hbar] \cdots [J^{(j)}_\theta + j\hbar] = 0.$$