PHYS 811: Assignment 3

1. The potential energy in the harmonic oscillator Lagrangian

\[ L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2, \]

where \( m \) is the mass and \( \omega \) is the fundamental frequency of the oscillator, belongs to the class of potentials for which the propagator can be rigorously represented in the form

\[ U(x,t,x',0) = A(t) \exp \left\{ \frac{im\omega}{2\hbar \sin(\omega t)} \left[ (x^2 + x'^2) \cos(\omega t) - 2xx' \right] \right\} \]

where \( A(t) \) is an unknown function (do not derive the expression for \( A(t) \)).

2. We know that given the eigenfunctions and the eigenvalues, we can construct the propagator as:

\[ U(x,t;x',t') = \sum_n \exp \left\{ -iE_n(t-t')/\hbar \right\} \psi_n(x)\psi_n^*(x') \]

Consider the reverse process (since the path integral approach gives \( U \) directly), for the case of the harmonic oscillator. The coefficient in Eq. (1) is \( A(t) = \sqrt{m\omega/2\pi} \). 

(a) Set \( x = x' = t' = 0 \). By expanding the left-hand side of Eq. (2) in powers of \( \alpha = \exp(-i\omega t) \) and writing the right-hand side in terms of \( \alpha \), you should find that \( E = \hbar \omega/2, 5\hbar \omega/2, 9\hbar \omega/2 \), etc. What happened to the levels in between?

(b) Now consider the extraction of the eigenfunctions. Let \( x = x' \) and \( t' = 0 \). Find \( E_0, E_1, |\psi_0(x)|^2 \), and \( |\psi_1(x)|^2 \).

3. Recall the derivation of the Schrödinger equation from the path-integral propagator for an infinitesimal increment of time \( \epsilon \) from time \( t = 0 \). In this derivation, the propagator was written as:

\[ U(x,\epsilon;x',0) = \sqrt{\frac{m}{2\pi\hbar\epsilon}} \exp \left\{ \frac{i}{\hbar} \left[ m(x - x')^2/(2\epsilon) - eV ((x + x')/2, 0) \right] \right\} \]

Note that although we chose the argument of \( V \) to be the midpoint \((x + x')/2\), it did not matter very much: any choice \( x + \alpha \eta \) (where \( \eta = x' - x \)) would have given the same result since the difference between the choices is of order \( \eta \epsilon = \epsilon^{3/2} \). All this was thanks to the factor \( \epsilon \) multiplying \( V \) in Eq. (3) and the fact that \( |\eta| \leq (2\pi\hbar\epsilon/m)^{1/2} \).

Consider now the case of a vector potential \( A \) which will bring in a factor

\[ \exp \left\{ \frac{i q\epsilon x - x'}{\hbar c} A(x + \alpha \eta) \right\} \equiv \exp \left\{ \frac{-i q\epsilon \eta}{\hbar c} A(x + \alpha \eta) \right\} , \]
where $q$ is the charge of particle and $c$ is velocity of light, to the propagator for one time slice. (We should really be using vectors for position and the vector potential, but the one-dimensional version will suffice for making the point here). Note that $\epsilon$ now gets canceled, in contrast to the scalar potential case. Thus, going to order $\epsilon$ to derive the Schrödinger equation means going to order $\eta^2$ in expanding the exponential. This will not only bring in an appropriate $A^2$ term, but will also make the answer sensitive to the argument of the linear term. Choose $\alpha = 1/2$ and verify that you get the one-dimensional version of the formula

$$H = \frac{1}{2m} \left( P \cdot P - \frac{q}{c} P \cdot A - \frac{q}{c} A \cdot P + \frac{q^2}{c^2} A \cdot A \right).$$