1. (i) Show that $P_1 = 3I/4 + S_1 \cdot S_2/h^2$ and $P_0 = I/4 - S_1 \cdot S_2/h^2$, where $S_i$ is spin 1/2 operator for particle $i$, are projection operators, i.e., obey $P_i P_j = \delta_{ij} P_j$.  Hint: Use $(A \cdot \sigma)(B \cdot \sigma) = (A \cdot B)I + i(A \times B) \cdot \sigma$.  

(ii) Show that these project into the spin-1 and spin-0 spaces in $1_2 \otimes 1_2 = 1 \oplus 0$.

2. (i) Show that equations

$$\begin{align*}
[J_\pm, T^q_k] &= \hbar \sqrt{(k \mp q)(k \pm q + 1)} T^{q \pm 1}_k \\
[J_z, T^q_k] &= \hbar q T^q_k
\end{align*}$$

(1)

(notice the spurious $\pm$ in front of square root in the text), where $T^q_k$ is $q$th component of a spherical tensor of rank $k$ and $J$ is an angular momentum operator, follow from the equation defining the tensor $T^q_k$

$$U[R] T^q_k U^\dagger[R] = \sum_{q'} D^{(k)}_{qq'} T^{q'}_{k'},$$

where $U[R]$ is a rotation operator and $D^{(k)}_{qq'}$ are matrix elements of this operator in the eigenbasis of $J^2$ and $J_z$, when one considers infinitesimal rotations.  Hint: $D^{(k)}_{qq'} = \langle kq'|I - i\epsilon \cdot J/h|kq\rangle$, where $|kq\rangle$ are eigenkets of $J^2$ and $J_z$.  Pick the infinitesimal rotation vector $\epsilon \hat{\theta}$ along the $\hat{x}_i$ directions.

(ii) Verify that the spherical tensor $V^\pm_1$ constructed out of a vector operator $V$ as $\mp(V_x \pm iV_y)/\sqrt{2}$ obeys Eqs. (1).

3. It is claimed that $\sum_q (-1)^q S^q_k T^{-q}_k$ is a scalar operator.

(i) For $k = 1$, verify that this is just $S \cdot T$.

(ii) Prove this expression for any $k$ by considering its response to a rotation.  Hint: $D^{(j)}_{-m,m'} = (-1)^{m-m'}(D^{(j)}_{m,m'})^*$.

4. (i) Consider a system whose angular momentum consists of two parts, $J_1$ and $J_2$, and whose magnetic moment is

$$\mu = \gamma_1 J_1 + \gamma_2 J_2.$$

In a state $|jm, j_1 j_2\rangle$, show, using the following equation

$$\langle \alpha' jm'|A^q_1|\alpha jm\rangle = \frac{\langle \alpha' jm'|J \cdot A|\alpha jm\rangle}{\hbar^2 j(j+1)} \langle jm'|J^q_1|jm\rangle,$$

where $A^q_1$ and $J^q_1$ are tensor components of operators $A$ and $J$, respectively, that

$$\langle \mu_x \rangle = \langle \mu_y \rangle = 0$$
and

\[ \langle \mu_z \rangle = m \hbar \left[ \frac{\gamma_1 + \gamma_2}{2} + \frac{(\gamma_1 - \gamma_2) j_1(j_1 + 1) - j_2(j_2 + 1)}{j(j + 1)} \right] . \]

(ii) Apply this to the problem of a proton \( g = 5.6 \) in a \( ^2P_{1/2} \) state and show that \( \langle \mu_z \rangle = \pm 0.27 \) nuclear magnetons.

(iii) For an electron in a \( ^2P_{1/2} \) state show that \( \langle \mu_z \rangle = \pm 1/3 \) Bohr magnetons.

5. Show that \( \langle jm|T_k^l|jm \rangle = 0 \) if \( k > 2j \).

6. Use \( \psi(x) = e^{-\alpha x^2} \) as a trial wave function in the variational method for a particle of mass \( m \) moving in one dimension in the potential \( V = m \omega^2 x^2 / 2 \). Find the value of \( \alpha = \alpha_0 \) that minimizes energy \( E \) and compare \( E(\alpha_0) \) with the exact solution.

7. Solve the variational problem for the \( l = 1 \) states of the electron in a potential \( V = -e^2/r \).

In your trial function, incorporate (i) correct behavior as \( r \to 0 \), appropriate for \( l = 1 \), (ii) correct number of nodes to minimize energy, (iii) correct behavior of wave function as \( r \to \infty \) in a Coulomb potential. Does it matter what \( m \) you choose for \( Y_l^m \)? Comment on the relation of the energy bound you obtain to the exact answer.