1. The trace of a matrix is defined as \( \text{Tr} \Omega = \sum \Omega_{ii} \). Show that (i) \( \text{Tr} \Omega \Lambda = \text{Tr} \Lambda \Omega \). (ii) \( \text{Tr} \Omega \Lambda \Theta = \text{Tr} \Lambda \Theta \Omega = \text{Tr} \Theta \Omega \Lambda \). (iii) \( \text{Tr} \Omega = \text{Tr} U^\dagger \Omega U \) where \( U \) is unitary.

2. Consider a particle tunneling through a rectangular potential barrier. Find the transmission coefficient \( T \) (i.e. the probability of penetrating the barrier and appearing on the other side).

3. Consider a particle in square-well potential

\[
V(x) = \begin{cases} 
0 & \text{for } |x| < a \\
V_0 & \text{for } |x| \geq a
\end{cases}
\]

Since, when \( V_0 \to \infty \), we have a box, let us guess what the lowering of the walls does to the states. First of all, all the bound states (which alone we are interested in), will have \( E \leq V_0 \).

Second, the wave functions of the low-lying levels will look like those of the particle in a box with the obvious difference that \( \psi \) will not vanish at the walls but instead spill out with an exponential tail. The eigenfunctions will still be even, odd, even, etc.

(i) Show that the even solutions have energies that satisfy the transcendental equation

\[
k \tan ka = \kappa
\]

while the odd ones will have energies that satisfy

\[
k \cot ka = -\kappa
\]

where \( k \) and \( i\kappa \) are the real and complex wave numbers inside and outside the well, respectively. Note that \( k \) and \( \kappa \) are related by

\[
k^2 + \kappa^2 = 2mV_0/\hbar^2.
\]

Verify that as \( V_0 \) tends to \( \infty \), we regain the levels in the box.

(ii) Equations (1) and (2) must be solved graphically. In the \( \alpha = ka \), \( \beta = \kappa a \) plane, imagine a circle that obeys Eq. (3). The bound states are then given by the intersection of the curve \( \alpha \tan a = \beta \) or \( \alpha \cot a = -\beta \) with the circle. (Remember that \( \alpha \) and \( \beta \) are positive).

(iii) Show that there is always one even solution and that there is no odd solution unless \( V_0 \geq \hbar^2 \pi^2 / 8ma^2 \). What is \( E \) when \( V_0 \) just meets this requirement? Note that every attractive potential in one dimension has at least one bound state.

4. Calculate the probabilities of transmission and reflection for a particle scattered off a potential \( V(x) = V_0 \delta(x) \). Use the time-dependent approach.

Hints:

\[
\int_{-\infty}^{\infty} e^{-ikx} \psi_1(x) \, dx = \sqrt{2\sqrt{\pi} \Delta} e^{-(k-k_0)^2/2} e^{ika}
\]
where
\[ \psi_1(x) \equiv \psi(x, 0) = \frac{1}{\sqrt{\sqrt{\pi} \Delta}} e^{ik_0(x+a)} e^{-(x+a)^2/2\Delta^2} \]
is the initial Gaussian wave packet. The free-particle time evolution of this packet is given by
\[ \psi(x, t) = \int U(x, t; x', 0) \psi(x', 0) \, dx' = \]
\[ = \sqrt{\pi} e^{-\frac{1}{2} \left[ (x + a - \frac{\hbar^2 k_0 t}{m})^2 \right]} \exp \left[ ik_0(x + a - \frac{\hbar k_0 t}{2m}) \right] \]
where \( U(x, t; x', 0) \) is the free-particle propagator
\[ U(x, t; x', 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} e^{-\frac{\hbar k^2 t}{2m}} \, dk = \]
\[ = \sqrt{\frac{m}{2\pi \hbar t}} \, e^{im(x-x')^2/2\hbar t} \]

5. Using the relation
\[ a|n> = \sqrt{n} |n-1> \]
where
\[ a = \sqrt{\frac{m \omega}{2\hbar}} X + i \sqrt{\frac{1}{2m \omega \hbar}} P \]
is the annihilation operator for a 1-D harmonic oscillator and \(|n>\) are the eigenvectors of the Hamiltonian for this system,
\[ <x|n> = \frac{1}{\sqrt{2^n n!}} (m \omega / \pi \hbar)^{1/4} e^{-y^2/2} H_n(y); \, \, y = \sqrt{m \omega / \hbar} x, \]
derive the recursion relation for Hermite’s polynomials
\[ \frac{dH_n(y)}{dy} = 2n H_{n-1}(y) \]

6. Consider two noninteracting particles of the same mass in a 1-D box. Assume that you can perform measurements of the energies and positions for these particles. Show that it is possible to determine whether the particles are bosons, fermions, or are distinguishable.