PHYS 607: Math Methods, Assignment 8
Due 11/10/03

1. Calculate

\[ \int_{-\infty}^{\infty} f(x)\delta(g(x))\,dx \]

where \( f \) and \( g \) are two arbitrary functions and \( \delta \) is the Dirac delta function. Prove all formulas that you use beyond the definition of \( \delta \).

2. Find properties of the distribution \( \delta(x^2) \).

3. This problem investigates the relation between closure and completeness.
   
   (a) Show that if a set of functions \( \{\phi_n(x)\}_{n=0}^{\infty} \) fulfills the closure relation
   
   \[ \delta(x - x') = \sum_{n=0}^{\infty} \phi_n(x)\phi_n(x') \]

   then this set is complete, i.e., it forms a basis in the space so that any function \( F(x) \) can be expanded in terms of \( \phi_n \)'s:

   \[ F(x) = \sum_{n=0}^{\infty} c_n\phi_n(x). \]

   The infinite sum in the equation above is defined as convergent in the mean:

   \[ \lim_{N \to \infty} \int |F(x) - \sum_{n=0}^{N} c_n\phi_n(x)|^2\,dx = 0. \]

   (b) In your proof of (a) you will encounter integrals \( \int F(x)\phi_n(x)\,dx \). How do you know that these integrals are finite?

4. Show that if the integral

\[ \int_{r_0}^{r} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \]

is independent of path, then there exists a function \( \varphi(\mathbf{r}) \) such that \( \mathbf{F} = \nabla \varphi \). Follow the (alternative) proof sketched in class, i.e., show first that the assumption implies that there exists a unique function \( G(x, y, z) \) such that

\[ G(\mathbf{r}) = \int_{r_0}^{r} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}', \]

where \( \mathbf{r} \) is any point in the space and \( \mathbf{r}_0 \) is an arbitrarily chosen fixed point, and, second, consider the increment of the function \( G \) upon \( (x, y, z) \rightarrow (x + \Delta x, y, z) \) to prove the theorem.