PHYS 607: Math Methods, Assignment 4
Due 10/10/03

1. (a) Proof the Baker-Hausdorff formula for matrices

\[ e^A Be^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots \]

(b) Since the proof is completely formal, it can apply as well to operators. Utilize therefore the Baker-Hausdorff formula for operators to evaluate

\[ e^{iHt/\hbar} X e^{-iHt/\hbar} \]

where \( X \) is the position operator and \( H \) is the free-particle Hamiltonian \( H = P^2/2m \), with \( P \) denoting the momentum operator. The commutator of \( X \) and \( P \) is \( [X, P] = i\hbar \).

2. Consider the following “four-vectors”, i.e., vectors with four components: \( \gamma = [\gamma^0, \gamma^1, \gamma^2, \gamma^3] \) whose elements are the four Dirac matrices and \( a = [a^0, a^1, a^2, a^3] \) where \( a^\mu \) are just numbers. The use of regular fonts for these vectors is intentional and in this way these vectors are distinguished from the vectors with three components denoted by the same letters but set in bold font. The scalar product of two four-vectors is defined as:

\[ a \cdot b = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \]

Notice that this can also be written, more conveniently, as:

\[ a \cdot b = a^\mu g_{\mu\nu} b^\nu = a^\mu g_{\mu\nu} b^\nu = a^\mu b_\mu = a_\mu b^\mu \]

where \( g_{\mu\nu} \) are the elements of the “metric tensor” \( (g^{00} = 1, g^{\mu\nu} = -1 \) for \( \mu = 1, 2, 3, \) and all other elements are zero). What kind of object is \( \gamma \cdot a \)? Since the scalar product amounts in this case to a sum of gamma matrices multiplied by numbers, \( \gamma \cdot a \) is just a 4x4 matrix. Show that

(a)

\[ \text{Tr} ((\gamma \cdot a)(\gamma \cdot b)) = 4a \cdot b \]

(b)

\[ \text{Tr} ((\gamma \cdot a)(\gamma \cdot b)(\gamma \cdot c)) = 0 \]
(c)

\[
\text{Tr} \left((\gamma \cdot a)(\gamma \cdot b)(\gamma \cdot c)(\gamma \cdot d)\right) = 4 \left((a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)\right)
\]

In addition to proving these relations, program them in Mathematica and make Mathematica check whether rhs is equal to lhs.


4. The Hamiltonian in the Dirac equation for a particle in a central potential \(V(r)\) is

\[
H = c\alpha \cdot p + \beta mc^2 + V(r)
\]

where \(c\) is the velocity of light, \(p\) is the momentum operator,

\[
\alpha = \begin{bmatrix}
0 & \sigma \\
\sigma & 0
\end{bmatrix}.
\]

\((\sigma = [\sigma_1, \sigma_2, \sigma_3])\), and

\[
\beta = \begin{bmatrix}
1_2 & 0 \\
0 & -1_2
\end{bmatrix}.
\]

It is known that \(H\), as well as the operator

\[
K = \beta \Sigma \cdot J - \frac{\hbar}{2} \beta,
\]

commutes with \(J\), the total angular momentum operator:

\[
J = L + \frac{\hbar}{2} \Sigma.
\]

In the equations above \(L\) denotes the orbital angular momentum operator and \(\Sigma = \alpha \gamma_5\) is the spin operator.

(a) Show that \(K\) commutes with \(H\).

(b) Let the eigenvalues of \(K\) and \(J^2\) be \(-\kappa \hbar^2\) and \(j(j+1)\hbar^2\), respectively. Show that

\[
\kappa^2 = \left(j + \frac{1}{2}\right)^2.
\]
5. Consider the matrix

\[ A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix} \]

(a) Find the eigenvalues of \( A \).

(b) Construct three orthonormal eigenvectors.

(c) Write out an explicit matrix \( S \) that diagonalizes \( A \) and show the corresponding diagonal form of \( A \).

6. Consider the matrix

\[ A = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -3 & 2 \\ -1 & 2 & 0 \end{pmatrix} \]

(a) Find the eigenvalues of \( A \).

(b) Construct three eigenvectors. Use explicitly Gram-Schmidt’s procedure to orthogonalize the eigenvectors. Can you find an orthogonal set of three eigenvectors?

(c) Is it possible to diagonalize \( A \) via a similarity transformation? If yes, find the matrix of such transformation.