Physics 207: Lecture 28

Announcements
• Final hwk assigned this week, final quiz next week
• Review session on Thursday May 19, 2:30 – 4:00pm, Here

Today’s Agenda
• Recap Angular Momentum
• Rotation about a fixed axis
  \[ \mathbf{L} = I \mathbf{\omega} \]
  • Example: Two disks
  • Student on rotating stool
• Angular momentum of a freely moving particle
  • Bullet hitting stick
  • Student throwing ball
• Comment about \( \tau = I \alpha \) (not true if \( I \) is changing!!)
• Vector considerations of angular momentum
  • Bike wheel and rotating stool

Angular momentum, \( \mathbf{L} \)

- \[ \tau_{\text{EXT}} = \frac{d\mathbf{L}}{dt} \]
  where \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \) and \( \tau_{\text{EXT}} = \mathbf{r} \times \mathbf{F}_{\text{EXT}} \)

- In the absence of external torques \( \tau_{\text{EXT}} = \frac{d\mathbf{L}}{dt} = 0 \)

Total angular momentum is conserved
Angular momentum of a rigid body about a fixed axis:

- Consider a rigid distribution of point particles rotating in the x-y plane around the z axis, as shown below. The total angular momentum around the origin is the sum of the angular momenta of each particle:

\[ \mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i = \sum_i m_i \mathbf{r}_i \mathbf{v}_i \hat{k} \quad \text{(since \( r_i \) and \( \mathbf{v}_i \) are perpendicular)} \]

We see that \( \mathbf{L} \) is in the z direction.

Using \( \mathbf{v}_i = \omega \mathbf{r}_i \), we get

\[ L = \sum_i m_i r_i^2 \omega \hat{k} \]

\[ \mathbf{L} = I \omega \] Analogue of \( \mathbf{p} = m \mathbf{v} \)

Angular momentum of a rigid body about a fixed axis:

- In general, for an object rotating about a fixed (z) axis we can write \( L_z = I \omega \)

- The direction of \( L_z \) is given by the right hand rule (same as \( \omega \)).

- We will omit the Z subscript for simplicity, and write \( L = I \omega \)
Example: Two Disks

- A disk of mass $M$ and radius $R$ rotates around the $z$ axis with angular velocity $\omega_i$. A second identical disk, initially not rotating, is dropped on top of the first. There is friction between the disks, and eventually they rotate together with angular velocity $\omega_f$.

- First realize that there are no external torques acting on the two-disk system.
  \[ \text{Angular momentum will be conserved!} \]

- Initially, the total angular momentum is due only to the disk on the bottom:
  \[ L_i = I_1 \omega_i = \frac{1}{2} MR^2 \omega_f \]
Example: Two Disks

- First realize that there are no external torques acting on the two-disk system.
  \[ \text{Angular momentum will be conserved!} \]

- Finally, the total angular momentum is due to both disks spinning:

\[
L_f = l_1 \omega_1 + l_2 \omega_2 = MR^2 \omega_f
\]

Example: Two Disks

- Since \( L_i = L_f \)
  \[
  \frac{1}{2} MR^2 \omega_i = MR^2 \omega_f
  \]

  \[
  \omega_f = \frac{1}{2} \omega_i
  \]

  An inelastic collision, since E is not conserved (friction)!
Example: Rotating Table

- A student sits on a rotating stool with his arms extended and a weight in each hand. The total moment of inertia is $I_i$ and he is rotating with angular speed $\omega_i$. He then pulls his hands in toward his body so that the moment of inertia reduces to $I_f$. What is his final angular speed $\omega_f$?

\[
L_i = I_i \omega_i \quad \text{and} \quad L_f = I_f \omega_f
\]

Example: Rotating Table...

- Again, there are no external torques acting on the student-stool system, so angular momentum will be conserved.

\[
\begin{align*}
L_i & = I_i \omega_i \\
L_f & = I_f \omega_f
\end{align*}
\]

\[
\frac{\omega_f}{\omega_i} = \frac{I_i}{I_f}
\]
A student sits on a freely turning stool and rotates with constant angular velocity $\omega_1$. She pulls her arms in, and due to angular momentum conservation her angular velocity increases to $\omega_2$. In doing this her kinetic energy:

(a) increases  (b) decreases  (c) stays the same

$L$ is conserved:

$I_2 < I_1 \implies K_2 > K_1 \quad \text{K increases!}$
Lecture 28, Act 1
Solution

- Since the student has to force her arms to move toward her body, she must be doing positive work!
- The work/kinetic energy theorem states that this will increase the kinetic energy of the system!

Angular Momentum of a Freely Moving Particle

- We have defined the angular momentum of a particle about the origin as $L = r \times p$
- This does not demand that the particle is moving in a circle!
  ➡️ We will show that this particle has a constant angular momentum!
Angular Momentum of a Freely Moving Particle...

- Consider a particle of mass $m$ moving with speed $v$ along the line $y = -d$. What is its angular momentum as measured from the origin $(0,0)$?

\[ r = \text{distance of closest approach} \]

\[ p = m v \]

\[ L = r \times p \]

The magnitude of the angular momentum is:

\[ |L| = |r \times p| = rp \sin \theta = p[r \sin \theta] = pd = p \times (\text{distance of closest approach}) \]

- Since $r$ and $p$ are both in the $x$-$y$ plane, $L$ will be in the $z$ direction (right hand rule): $L_z = pd$
Angular Momentum of a Freely Moving Particle...

- So we see that the direction of $L$ is along the $z$ axis, and its magnitude is given by $L_z = pd = mvd$.
- $L$ is clearly conserved since $d$ is constant (the distance of closest approach of the particle to the origin) and $p$ is constant (momentum conservation).

![Diagram of a particle with angular momentum](image)

Example: Bullet hitting stick

- A uniform stick of mass $M$ and length $D$ is pivoted at the center. A bullet of mass $m$ is shot through the stick at a point halfway between the pivot and the end. The initial speed of the bullet is $v_1$, and the final speed is $v_2$.
  - What is the angular speed $\omega_f$ of the stick after the collision? (Ignore gravity)
Example: Bullet hitting stick...

- Conserve angular momentum around the pivot (z) axis!
- The total angular momentum before the collision is due only to the bullet (since the stick is not rotating yet).

\[ L_i = p \times \text{(distance of closest approach)} = mv_1 \frac{D}{4} \]

Example: Bullet hitting stick...

- Conserve angular momentum around the pivot (z) axis!
- The total angular momentum after the collision has contributions from both the bullet and the stick.

\[ L_f = mv_2 \frac{D}{4} + I \omega_F \]

where \( I \) is the moment of inertia of the stick about the pivot.
Example: Bullet hitting stick...

- Set \( L_i = L_f \) using \( I = \frac{1}{12} MD^2 \)

\[
mv_1 \frac{D}{4} = mv_2 \frac{D}{4} + \frac{1}{12} MD^2 \omega_f
\]

\[
\Rightarrow \quad \omega_f = \frac{3m}{MD} (v_1 - v_2)
\]

Example: Throwing ball from stool

- A student sits on a stool which is free to rotate. The moment of inertia of the student plus the stool is \( I \). She throws a heavy ball of mass \( M \) with speed \( v \) such that its velocity vector passes a distance \( d \) from the axis of rotation.

What is the angular speed \( \omega_f \) of the student-stool system after she throws the ball?
Example: Throwing ball from stool...

- Conserve angular momentum (since there are no external torques acting on the student-stool system):
  \( L_i = 0 \)
  \( L_f = 0 = I \omega_f - Mvd \)
  \[ \omega_f = \frac{Mvd}{I} \]

![Diagram of throwing ball from stool](image)

top view: initial  final

Review...

- A freely moving particle has a definite angular momentum about any axis.
- If no torques are acting on the particle, its angular momentum will be conserved.
- In the example below, the direction of \( L \) is along the \( z \) axis, and its magnitude is given by \( L_z = pd = mvd \).

![Diagram of particle](image)
Angular momentum

- Two different spinning disks have the same angular momentum, but disk 1 has more kinetic energy than disk 2.
  ➔ Which one has the biggest moment of inertia?

(a) disk 1       (b) disk 2       (c) not enough info

Solution

\[ K = \frac{1}{2} I_1 \omega_1^2 = \frac{1}{2I} L^2 \omega_1^2 = \frac{1}{2I} L^2 \] (using \( L = I \omega \))

If they have the same \( L \), the one with the biggest \( I \) will have the smallest kinetic energy.
When does $\tau = I \alpha$ not work?

- Last time we showed that $\tau = \frac{dL}{dt}$
- This is the fundamental equation for understanding rotation.
- If we write $L = I\omega$, then

$$\frac{dL}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} + \omega \frac{dI}{dt} = I\alpha + \omega \frac{dI}{dt}$$

$$\tau = I\alpha + \omega \frac{dI}{dt}$$

We can’t assume $\tau = I\alpha$ when the moment of inertia is changing!

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When does $\tau = I\alpha$ not work?

$$\tau = I\alpha + \omega \frac{dI}{dt}$$

Now suppose $\tau = 0$:

$$I\alpha + \omega \frac{dI}{dt} = 0$$

$$\alpha = -\frac{\omega}{I} \frac{dI}{dt}$$

So in this case we can have an $\alpha$ without an external torque!
Example...

- A puck in uniform circular motion will experience rotational acceleration if its moment of inertia is changed.

- Changing the radius changes the moment of inertia, but produces no torque since the force of the string is along the radial direction. (since $\mathbf{r} \times \mathbf{F} = 0$)

$$I_1 > I_2$$

$$\omega_2 > \omega_1$$

The puck accelerates without external torque!!

Lecture 28, Act 3
Rotations

- A puck slides in a circular path on a horizontal frictionless table. It is held at a constant radius by a string threaded through a frictionless hole at the center of the table. If you pull on the string such that the radius decreases by a factor of 2, by what factor does the angular velocity of the puck increase?

(a) 2    (b) 4    (c) 8
Lecture 28, Act 3
Solution

- Since the string is pulled through a hole at the center of rotation, there is no torque: Angular momentum is conserved.

\[ L_1 = I_1 \omega_1 = mR^2 \omega_1 = L_2 = I_2 \omega_2 = m \left( \frac{R}{2} \right)^2 \omega_2 \]

\[ mR^2 \omega_1 = m \frac{1}{4} R^2 \omega_2 \]

\[ \omega_1 = \frac{1}{4} \omega_2 \quad \Rightarrow \quad \omega_2 = 4 \omega_1 \]

Review: Angular Momentum

- \[ \tau_{\text{EXT}} = \frac{dL}{dt} \]
  where \[ L = r \times p \]
  and \[ \tau_{\text{EXT}} = r \times F_{\text{EXT}} \]

- In the absence of external torques \[ \tau_{\text{EXT}} = \frac{dL}{dt} = 0 \]

  Total angular momentum is conserved

- This is a vector equation.
- Valid for individual components.
Review...

- In general, for an object rotating about a fixed \((z)\) axis we can write \(L_z = I \omega\).

- The direction of \(L_z\) is given by the right hand rule (same as \(\omega\)).

Angular momentum is a vector!

Demo: Turning the bike wheel.

- A student sits on the rotatable stool holding a bicycle wheel that is spinning in the horizontal plane. She flips the rotation axis of the wheel \(180^\circ\), and finds that she herself starts to rotate.

\(\Downarrow\) What’s going on?
Turning the bike wheel...

- Since there are no external torques acting on the student-stool system, angular momentum is conserved.
  - Initially: $L_{\text{ini}} = L_{W,I}$
  - Finally: $L_{\text{fin}} = L_{W,F} + L_S$

Lecture 28, Act 4
Rotations

- A student is initially at rest on a rotatable chair, holding a wheel spinning as shown in (1). He turns it over and starts to rotate (2). If he keeps twisting, turning the wheel over again (3), his rotation will:
  - (a) stop
  - (b) double
  - (c) stay the same
Lecture 28, Act 4
Solution

\[ L_{\text{NET}} \]

\[ L_{\text{W}} \]

\[ L_{\text{NET}} \]

\[ L_{\text{NET}} \]

\[ L_{\text{S}} \]

\[ L_{\text{W}} \]

not turning