Physics 207: Lecture 25

Announcements

• Exam 3 review session: TODAY from 8:00 – 9:30 pm; here
• Exam 3 on Friday, May 6th
  - Bring photo-ID, scientific calculator, pencil, eraser
  - Topics from Chapter 8, 9, and 10

Today’s Agenda

• Rotation Recap
• Many body dynamics examples
  ➜ Rolling down an incline
  ➜ Bowling ball: sliding to rolling
  ➜ Atwood’s Machine with a massive pulley
• Torque due to gravity

Torque

• Recall the definition of torque:

\[ \tau = rF \theta \]
\[ = r F \sin \phi \]
\[ = r \sin \phi F \]

\[ \tau = rpF \]

\[ r_p = \text{"distance of closest approach"} \]

• More generally \( \vec{\tau} = \vec{r} \times \vec{F} \)
Work & Power

- The work done by a torque $\tau$ acting through a displacement $\theta$ is given by:
  \[
  W = \tau \theta
  \]

- The power provided by a constant torque is therefore given by:
  \[
  P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau \omega
  \]

Recap of Rotation

- About a fixed rotation axis, you can always write $\tau = I \alpha$, where $\tau$ is the torque, $I$ is the moment of inertia, and $\alpha$ is the angular acceleration.
- For discrete point particles, $I = \sum m_i r_i^2$.
- The parallel axis theorem lets you calculate the moment of inertia about an axis parallel to an axis through the CM if you know $I_{CM}$:
  \[
  I_{PARALLEL} = I_{CM} + MD^2
  \]
- If the object is accelerating, we can still use $\tau = I \alpha$ provided that we are considering rotations about an axis through the CM.
**Work & Kinetic Energy:**

- Recall the Work/Kinetic Energy Theorem: \( \Delta K = W_{NET} \)

- This is true in general, and hence applies to rotational motion as well as linear motion.

- So for an object that rotates about a fixed axis:

  \[
  \Delta K = \frac{1}{2} (\omega_f^2 - \omega_i^2) = W_{NET}
  \]

- For objects that are translating and rotating

  \[
  \Delta K = \frac{1}{2} I (\omega_f^2 - \omega_i^2) + \frac{1}{2} M \left( V_{cm,f}^2 - V_{cm,i}^2 \right) = W_{NET}
  \]

**Falling weight & pulley**

- A mass \( m \) is hung by a string that is wrapped around a pulley of radius \( R \) attached to a heavy flywheel. The moment of inertia of the pulley + flywheel is \( I \). The string does not slip on the pulley.

  \[\begin{align*}
  \text{Starting at rest, how long does it take} \\
  \text{for the mass to fall a distance } L.
  \end{align*}\]
Falling weight & pulley...

- For the hanging mass use \( F = ma \)
  \( \iff mg - T = ma \)
- For the pulley + flywheel use \( \tau = I\alpha \)
  \( \iff \tau = TR = I\frac{a}{R} \)
- Realize that \( a = \alpha R \iff TR = I\frac{a}{R} \)
- Now solve for \( a \) using the above equations.

\[
a = \left( \frac{mR^2}{mR^2 + I} \right) g
\]

Falling weight & pulley...

- Using 1-D kinematics (Lecture 2) we can solve for the time required for the weight to fall a distance \( L \):

\[
L = \frac{1}{2} at^2 \iff t = \sqrt{\frac{2L}{a}}
\]

where \( a = \left( \frac{mR^2}{mR^2 + I} \right) g \)
Rolling

- An object with mass $M$, radius $R$, and moment of inertia $I$ rolls without slipping down a plane inclined at an angle $\theta$ with respect to horizontal. What is its acceleration?

- Consider CM motion and rotation about the CM separately when solving this problem

\[ F_{\text{NET}} = MA_{CM} \]

In the $x$ direction: $Mg \sin \theta - f = MA$

Rolling...

- Static friction $f$ causes rolling. It is an unknown, so we must solve for it.

- First consider the free body diagram of the object and use $F_{\text{NET}} = MA_{CM}$:

  \[ Rf = I \frac{A}{R} \quad \Rightarrow \quad f = I \frac{A}{R^2} \]
Rolling...

- We have two equations: 
  \[ Mg \sin \theta - f = mA \quad f = I \frac{A}{R^2} \]

- We can combine these to eliminate \( f \):
  \[
  A = g \frac{MR^2 \sin \theta}{MR^2 + I}
  \]

For a sphere:
  \[
  A = g \frac{MR^2 \sin \theta}{MR^2 + \frac{2}{5}MR^2} = 5 \frac{g \sin \theta}{7}
  \]

Sliding to Rolling

- A bowling ball of mass \( M \) and radius \( R \) is thrown with initial velocity \( v_0 \). It is initially not rotating. After sliding with kinetic friction along the lane for a distance \( D \) it finally rolls without slipping and has a new velocity \( v_f \). The coefficient of kinetic friction between the ball and the lane is \( \mu \).

> What is the final velocity, \( v_f \), of the ball?
Sliding to Rolling...

- While sliding, the force of friction will accelerate the ball in the -x direction: \( F = -\mu Mg = Ma \) so \( a = -\mu g \)
- The speed of the ball is therefore \( v = v_0 - \mu gt \) (a)
- Friction also provides a torque about the CM of the ball. Using \( \tau = I\alpha \) and remembering that \( I = \frac{2}{5}MR^2 \) for a solid sphere about an axis through its CM:

\[
\tau = \mu MgR = \frac{2}{5}MR^2 \alpha \Rightarrow \quad \alpha = \frac{5\mu g}{2R} \Rightarrow \quad \omega = \omega_0 + \alpha t = \frac{5\mu g}{2R} t \quad (b)
\]

\[
\begin{align*}
\tau &= \mu MgR = \frac{2}{5}MR^2 \alpha \\
\omega &= \frac{5\mu g}{2R} t \\
\end{align*}
\]

\[
\begin{array}{c}
\tau = \mu MgR = \frac{2}{5}MR^2 \alpha \\
\omega = \frac{5\mu g}{2R} t \\
\end{array}
\]

We have two equations:

\( v = v_0 - \mu gt \) (a) \quad \omega = \frac{5\mu g}{2R} t \quad (b)

Using (b) we can solve for \( t \) as a function of \( \omega \):

\[ t = \frac{2R\omega}{5\mu g} \]

Plugging this into (a) and using \( v_t = \omega R \) (the condition for rolling without slipping):

\[
\begin{align*}
v_f &= \frac{5}{7}v_0 \\
\end{align*}
\]

Doesn't depend on \( \mu, M, g \)!!

\[
\begin{array}{c}
v_f = \frac{5}{7}v_0 \\
\end{array}
\]

\[
\begin{array}{c}
\omega = \frac{5\mu g}{2R} t \\
\end{array}
\]
Lecture 25, Act 1
Rotations

• A bowling ball (uniform solid sphere) rolls along the floor without slipping.

What is the ratio of its rotational kinetic energy to its translational kinetic energy?

(a) \( \frac{1}{5} \)  
(b) \( \frac{2}{5} \)  
(c) \( \frac{1}{2} \)

Recall that \( I = \frac{2}{5}MR^2 \) for a solid sphere about an axis through its CM:

Lecture 25, Act 1
Solution

• The total kinetic energy is partly due to rotation and partly due to translation (CM motion).

\[ K = \frac{1}{2} I \omega^2 + \frac{1}{2} MV^2 \]

rotational  
translational

K  
K
Lecture 25, Act 1
Solution

\[ K = \frac{1}{2} I \omega^2 + \frac{1}{2} MV^2 \]

Since it rolls without slipping: \( \omega = \frac{V}{R} \)

rotational \hspace{1cm} Translational
\[ K \hspace{1cm} K \]

\[
K_{\text{ROT}} = \frac{\frac{1}{2} I \omega^2}{\frac{1}{2} MV^2} = \frac{\left(\frac{2}{5} MR^2\right)}{MV^2} = \frac{2}{5}
\]

Lecture 25, Act 2
Rotations

- A ball and box have the same mass and are moving with the same velocity across a horizontal floor. The ball rolls without slipping and the box slides without friction. They encounter an upward slope in the floor. Which one makes it farther “up the hill” before stopping?

(a) ball \hspace{1cm} (b) box \hspace{1cm} (c) same
Lecture 25, Act 2
Solution

- The ball and box will stop when their initial kinetic energies have been converted to gravitational potential energy $(mgH)$.
- The initial kinetic energy of the box is $K = \frac{1}{2}mv^2$
- The initial kinetic energy of the ball is $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$ \textbf{bigger}

Since the ball has more initial kinetic energy, it will go higher!
Atwoods Machine with Massive Pulley:

- A pair of masses are hung over a massive disk-shaped pulley as shown. 
  - Find the acceleration of the blocks.
- For the hanging masses use $F = ma$
  - $-m_1g + T_1 = -m_1a$
  - $-m_2g + T_2 = m_2a$
- For the pulley use $\tau = l\alpha = \frac{1}{2}ma$
  - $T_1R - T_2R = \frac{1}{2}MRa$
  
  (Since $l = \frac{1}{2}MR^2$ for a disk)

Atwoods Machine with Massive Pulley...

- We have three equations and three unknowns ($T_1, T_2, a$). Solve for $a$.
  - $-m_1g + T_1 = -m_1a \quad (1)$
  - $-m_2g + T_2 = m_2a \quad (2)$
  - $T_1 - T_2 = \frac{1}{2}Ma \quad (3)$
  - $a = \left( \frac{m_1 - m_2}{m_1 + m_2 + M/2} \right)g$
Torque due to Gravity

- As we now know $\sum r_i = l\alpha$ where $\tau_i = r_i \times F_i$
- Take the rotation axis to be along the $z$ direction (as usual) and recall that
  $\tau_i = \tau_{z,i} = r_{x,i} F_{y,i} - F_{x,i} r_{y,i} = x_i (-m_i g) - 0$
  (also obtain via “closest approach” method)

$$\sum r_i = -g \sum m_i x_i = -g M x_{cm}$$

So: $\tau = -M g x_{cm}$

Where: $M = \sum m_i$

Torque due to Gravity...

- But this is the same expression we would get if we were to find the CM...
Torque due to Gravity...

- ...and assume that all of the mass was located there!

- So for the purpose of figuring out the torque due to gravity, you can treat an object as though all of its mass were located at the center of mass.

\[ \tau_{NET} = -Mg x_{cm} \]
\[ M = \sum_i m_i \]

Example problem: Yo-yo

- Problem 10.77 done on the board