Physics 207: Lecture 24

Announcements

• No labs next week, May 2 – 5
• Exam 3 review session: Wed, May 4 from 8:00 – 9:30 pm; here

Today’s Agenda

• Recap:
  ←Rotational dynamics and torque
• Work and energy with example
• Many body dynamics examples
  ←Weight and massive pulley
  ←Rolling and sliding examples
  ←Rotation around a moving axis: Puck on ice
  ←Rolling down an incline
  ←Bowling ball: sliding to rolling
  ←Atwood’s Machine with a massive pulley

Review: Torque and Angular Acceleration

\[ \tau_{\text{NET}} = I \alpha \]

• This is the rotational analogue
  of \( F_{\text{NET}} = ma \)
• Torque is the rotational analogue of force:
  ←The amount of “twist” provided by a force.
• Moment of inertia \( I \) is the rotational analogue of mass
  ←If \( I \) is big, more torque is required to achieve a given angular acceleration.
Torque

- Recall the definition of torque:

\[ \tau = rF \sin \phi \]
\[ = r F \sin \phi \]
\[ = r \sin \phi \]
\[ \tau = r_p F \]

- Equivalent definitions!

\[ r_p = \text{"distance of closest approach"} \]

- So if \( \phi = 0^\circ \), then \( \tau = 0 \)

- And if \( \phi = 90^\circ \), then \( \tau = \text{maximum} \)
Lecture 23, Act 3
Torque

- In which of the cases shown below is the torque provided by the applied force about the rotation axis biggest? In both cases the magnitude and direction of the applied force is the same.

(a) case 1  
(b) case 2  
(c) same

Lecture 23, Act 3
Solution

- Torque = \( F \times \text{(distance of closest approach)} \)
  - The applied force is the same.
  - The distance of closest approach is the same.

\[ \text{Torque is the same!} \]
Torque and the Right Hand Rule:

- The right hand rule can tell you the direction of torque:
  - Point your hand along the direction from the axis to the point where the force is applied.
  - Curl your fingers in the direction of the force.
  - Your thumb will point in the direction of the torque.

The Cross Product

- We can describe the vectorial nature of torque in a compact form by introducing the "cross product".
  - The cross product of two vectors is a third vector:

\[
A \times B = C
\]

- The length of \( C \) is given by:
  \[
  C = AB \sin \phi
  \]

- The direction of \( C \) is perpendicular to the plane defined by \( A \) and \( B \), and in the direction defined by the right hand rule.
The Cross Product

- Cartesian components of the cross product:

\[ C = A \times B \]

\[ C_x = A_y B_z - B_y A_z \]

\[ C_y = A_z B_x - B_z A_x \]

\[ C_z = A_x B_y - B_x A_y \]

Note: \( B \times A = -A \times B \)

Torque & the Cross Product:

- So we can define torque as:

\[ \tau = r \times F \]

\[ = rF \sin \phi \]

\[ \tau_x = r_y F_z - F_y r_z = y F_z - F_y z \]

\[ \tau_y = r_z F_x - F_z r_x = z F_x - F_z x \]

\[ \tau_z = r_x F_y - F_x r_y = x F_y - F_x y \]
Comment on $\tau = I\alpha$

- When we write $\tau = I\alpha$ we are really talking about the $z$ component of a more general vector equation. (Recall that we normally choose the $z$-axis to be the rotation axis.)

$$\tau_z = I_z \alpha_z$$

- We usually omit the $z$ subscript for simplicity.

Example

- To loosen a stuck nut, a (stupid) man pulls at an angle of $45^\circ$ on the end of a $50 \text{ cm}$ wrench with a force of $200 \text{ N}$.
  - What is the magnitude of the torque on the nut?
  - If the nut suddenly turns freely, what is the angular acceleration of the wrench? (The wrench has a mass of 3 kg, and its shape is that of a thin rod.)
Example

- Torque \( \tau = LF \sin \phi = (0.5 \text{ m})(200 \text{ N})(\sin 45) = 70.7 \text{ Nm} \)

- If the nut turns freely, \( \tau = I\alpha \)
  - We know \( \tau \) and we want \( \alpha \), so we need to figure out \( I \).

\[
I = \frac{1}{3} ML^2 = \frac{1}{3} (3 \text{ kg})(0.5 \text{ m})^2 = 0.25 \text{ kgm}^2
\]

So \( \alpha = \tau / I = (70.7 \text{ Nm}) / (0.25 \text{ kgm}^2) \)

\[\alpha = 283 \text{ rad/s}^2\]

Work

- Consider the work done by a force \( F \) acting on an object constrained to move around a fixed axis. For an infinitesimal angular displacement \( d\theta \):

\[
\sum dW = F \cdot dr = FR \ d\theta \cos(\beta)
\]

\[
= FR \ d\theta \cos(90-\phi)
\]

\[
= FR \ d\theta \sin(\phi)
\]

\[
= FR \ sin(\phi) \ d\theta
\]

\[dW = \tau \ d\theta\]

- We can integrate this to find: \( W = \tau \theta \)
- Analogue of \( W = F \cdot dr \)
- \( W \) will be negative if \( \tau \) and \( \theta \) have opposite signs!
Work & Power

- The work done by a torque $\tau$ acting through a displacement $\theta$ is given by:

$$W = \tau \theta$$

- The power provided by a constant torque is therefore given by:

$$P = \frac{dW}{dt} = \frac{d\theta}{dt} = \tau \omega$$

Work & Kinetic Energy:

- Recall the Work/Kinetic Energy Theorem: $\Delta K = W_{\text{NET}}$

- This is true in general, and hence applies to rotational motion as well as linear motion.

- So for an object that rotates about a fixed axis:

$$\Delta K = \frac{1}{2} I (\omega_f^2 - \omega_i^2) = W_{\text{NET}}$$
Example: Disk & String

- A massless string is wrapped 10 times around a disk of mass $M = 40 \text{ g}$ and radius $R = 10 \text{ cm}$. The disk is constrained to rotate without friction about a fixed axis though its center. The string is pulled with a force $F = 10 \text{ N}$ until it has unwound. (Assume the string does not slip, and that the disk is initially not spinning).

How fast is the disk spinning after the string has unwound?

Disk & String...

- The work done is $W = \tau \theta$
  - The torque is $\tau = RF$ (since $\phi = 90^\circ$)
  - The angular displacement $\theta$ is $2\pi \text{ rad/rev} \times 10 \text{ rev}$.

- So $W = (.1 \text{ m})(10 \text{ N})(20\pi \text{ rad}) = 62.8 \text{ J}$
Disk & String...

\[ W_{NET} = W = 62.8 \text{ J} = \Delta K = \frac{1}{2} I \omega^2 \]

Recall that \( I \) for a disk about its central axis is given by:

\[ I = \frac{1}{2} MR^2 \]

So \( \Delta K = \frac{1}{2} \left( \frac{1}{2} MR^2 \right) \omega^2 = W \)

\[ \omega = \sqrt{\frac{4W}{MR^2}} = \sqrt{\frac{4(62.8 \text{ J})}{(0.04 \text{ kg})(1)^2}} \implies \omega = 792.5 \text{ rad/s} \]

Lecture 23, Act 4
Work & Energy

- Strings are wrapped around the circumference of two solid disks and pulled with identical forces for the same distance. Disk 1 has a bigger radius, but both have the same moment of inertia. Both disks rotate freely around axes though their centers, and start at rest.

\( \Leftarrow \) Which disk has the biggest angular velocity after the pull?

(a) disk 1
(b) disk 2
(c) same
Lecture 23, Act 4
Solution

- The work done on both disks is the same!
  \[ W = Fd \]
- The change in kinetic energy of each will therefore also be the same since \( W = \Delta K \).

But we know \( \Delta K = \frac{1}{2}I\omega^2 \)

So since \( I_1 = I_2 \)

\[ \omega_1 = \omega_2 \]

Lecture 24, Act 1
Rotations

- Two wheels can rotate freely about fixed axles through their centers. The wheels have the same mass, but one has twice the radius of the other.
- Forces \( F_1 \) and \( F_2 \) are applied as shown. What is \( F_2 / F_1 \) if the angular acceleration of the wheels is the same?

(a) 1
(b) 2
(c) 4
Lecture 24, Act 1
Solution

We know \( \tau = I \alpha \)

but \( \tau = FR \) and \( I = mR^2 \)

so \( FR = mR^2 \alpha \)
\( F = mR \alpha \)

\( \Rightarrow \frac{F_2}{F_1} = \frac{mR_2 \alpha}{mR_1 \alpha} = \frac{R_2}{R_1} \)

Since \( R_2 = 2R_1 \)

\( \Rightarrow \frac{F_2}{F_1} = 2 \)

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Falling weight & pulley

- A mass \( m \) is hung by a string that is wrapped around a pulley of radius \( R \) attached to a heavy flywheel. The moment of inertia of the pulley + flywheel is \( I \). The string does not slip on the pulley.

\( \Leftarrow \) Starting at rest, how long does it take for the mass to fall a distance \( L \).
Falling weight & pulley...

- For the hanging mass use $F = ma$
  \[ mg - T = ma \]
- For the pulley + flywheel use $\tau = I\alpha$
  \[ \tau = TR = I\alpha \]
- Realize that $a = \alpha R \Rightarrow TR = \frac{a}{R}$
- Now solve for $a$ using the above equations.

\[
a = \left( \frac{mR^2}{mR^2 + I} \right) g
\]

Falling weight & pulley...

- Using 1-D kinematics (Lecture 2) we can solve for the time required for the weight to fall a distance $L$:

\[
L = \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2L}{a}}
\]

where \[ a = \left( \frac{mR^2}{mR^2 + I} \right) g \]
Rotation around a moving axis.

- A string is wound around a puck (disk) of mass $M$ and radius $R$. The puck is initially lying at rest on a frictionless horizontal surface. The string is pulled with a force $F$ and does not slip as it unwinds.

What length of string $L$ has unwound after the puck has moved a distance $D$?

Rotation around a moving axis...

- The CM moves according to $F = MA \implies A = \frac{F}{M}$
- The distance moved by the CM is thus $D = \frac{1}{2} At^2 = \frac{F}{2M} t^2$
- The disk will rotate about its CM according to $\tau = I\alpha \implies \alpha = \frac{\tau}{I} = \frac{RF}{\frac{1}{2}MR^2} = \frac{2F}{MR}$
- So the angular displacement is $\theta = \frac{1}{2} \alpha t^2 = \frac{F}{MR} t^2$
Rotation around a moving axis...

- So we know both the distance moved by the CM and the angle of rotation about the CM as a function of time:

\[ D = \frac{F}{2M} t^2 \quad (a) \]
\[ \theta = \frac{F}{MR} t^2 \quad (b) \]

Divide (b) by (a):

\[ \frac{\theta}{D} = \frac{2}{R} \Rightarrow R\theta = 2D \]

The length of string pulled out is \( L = R\theta \):

![Diagram of rotation around a moving axis]

Comments on CM acceleration:

- We just used \( \tau = I\alpha \) for rotation about an axis through the CM even though the CM was accelerating!
  - The CM is not an inertial reference frame! Is this OK?? (After all, we can only use \( F = ma \) in an inertial reference frame).

- YES! We can always write \( \tau = I\alpha \) for an axis through the CM.
  - This is true even if the CM is accelerating.
  - We will prove this when we discuss angular momentum!
Rolling

- An object with mass $M$, radius $R$, and moment of inertia $I$ rolls without slipping down a plane inclined at an angle $\theta$ with respect to horizontal. What is its acceleration?

- Consider CM motion and rotation about the CM separately when solving this problem (like we did with the last problem)...

Rolling...

- Static friction $f$ causes rolling. It is an unknown, so we must solve for it.

- First consider the free body diagram of the object and use $F_{\text{NET}} = MA_{\text{CM}}$:

  In the $x$ direction: $Mg \sin \theta - f = MA$

- Now consider rotation about the CM and use $\tau = I \alpha$ realizing that $\tau = Rf$ and $A = \alpha R$

  $Rf = I \frac{A}{R} \Leftrightarrow f = \frac{I A}{R^2}$
Rolling...

- We have two equations:
  \[ Mg \sin \theta - f = mA \quad f = I \frac{A}{R^2} \]

- We can combine these to eliminate \( f \):
  \[ A = g \frac{MR^2 \sin \theta}{MR^2 + I} \]

For a sphere:
\[ A = g \frac{MR^2 \sin \theta}{MR^2 + \frac{2}{5}MR^2} = \frac{5}{7} g \sin \theta \]

Lecture 24, Act 2
Rotations

- Two uniform cylinders are machined out of solid aluminum. One has twice the radius of the other.
  ❯ If both are placed at the top of the same ramp and released, which is moving faster at the bottom?

(a) bigger one
(b) smaller one
(c) same
Lecture 24, Act 2
Solution

Consider one of them. Say it has radius \( R \), mass \( M \) and falls a height \( H \).

Energy conservation: \(- \Delta U = \Delta K\)  \(\Rightarrow\)  \(MgH = \frac{1}{2} I \omega^2 + \frac{1}{2} MV^2\)

but \( l = \frac{1}{2} MR^2 \) and \( \omega = \frac{V}{R} \)

\(\Rightarrow\)  \(MgH = \frac{1}{2} \left( \frac{1}{2} MR^2 \right) \frac{V^2}{R^2} + \frac{1}{2} MV^2\)

\(\Rightarrow\)  \(MgH = \frac{1}{4} MV^2 + \frac{1}{2} MV^2 = \frac{3}{4} MV^2\)

So: \(MgH = \frac{3}{4} MV^2\)  \(\Rightarrow\)  \(gH = \frac{3}{4} V^2\)

\(\Rightarrow\)  \(V = \sqrt{\frac{4}{3} gH}\)

So, (c) does not depend on size, as long as the shape is the same!!
Sliding to Rolling

- A bowling ball of mass $M$ and radius $R$ is thrown with initial velocity $v_0$. It is initially not rotating. After sliding with kinetic friction along the lane for a distance $D$ it finally rolls without slipping and has a new velocity $v_f$. The coefficient of kinetic friction between the ball and the lane is $\mu$.

What is the final velocity, $v_f$, of the ball?

\[ v_f = \omega R \]

\[ f = \mu Mg \]

Sliding to Rolling...

- While sliding, the force of friction will accelerate the ball in the -x direction: $F = -\mu Mg = Ma$ so $a = -\mu g$
- The speed of the ball is therefore $v = v_0 - \mu gt$ (a)
- Friction also provides a torque about the CM of the ball. Using $\tau = I\alpha$ and remembering that $I = \frac{2}{5}MR^2$ for a solid sphere about an axis through its CM:

\[ \tau = \mu MgR = \frac{2}{5}MR^2\alpha \quad \Rightarrow \quad \alpha = \frac{5\mu g}{2R} \quad \Rightarrow \quad \omega = \omega_0 + \alpha t = \frac{5\mu gt}{2R} \quad (b) \]
Sliding to Rolling...

- We have two equations: \( v = v_0 - \mu gt \) (a) \( \omega = \frac{5\mu g t}{2R} \) (b)
- Using (b) we can solve for \( t \) as a function of \( \omega \): \( t = \frac{2R\omega}{5\mu g} \)
- Plugging this into (a) and using \( v_f = \omega R \) (the condition for rolling without slipping):

\[
\begin{align*}
\omega &= \frac{5}{7} \omega_0 \\
\Rightarrow v_f &= \frac{5}{7} v_0 \\
\text{Doesn't depend on } \mu, M, g \text{!!}
\end{align*}
\]

Lecture 24, Act 3
Rotations

- A bowling ball (uniform solid sphere) rolls along the floor without slipping.
- What is the ratio of its rotational kinetic energy to its translational kinetic energy?

(a) \( \frac{1}{5} \) \hspace{1cm} (b) \( \frac{2}{5} \) \hspace{1cm} (c) \( \frac{1}{2} \)

Recall that \( I = \frac{2}{5} MR^2 \) for a solid sphere about an axis through its CM:
Lecture 24, Act 3
Solution

- The total kinetic energy is partly due to rotation and partly due to translation (CM motion).

\[ K = \frac{1}{2} I \omega^2 + \frac{1}{2} MV^2 \]

rotational \hspace{1cm} translational

\[ K \hspace{1cm} K \]

Since it rolls without slipping: \( \omega = \frac{V}{R} \)

\[ K_{\text{ROT}} = \frac{\frac{1}{2} I \omega^2}{\frac{1}{2} MV^2} = \frac{(\frac{2}{5} MR^2) \frac{V^2}{R^2}}{MV^2} = \frac{2}{5} \]
Atwoods Machine with Massive Pulley:

- A pair of masses are hung over a massive disk-shaped pulley as shown.
  
- Find the acceleration of the blocks.

- For the hanging masses use $F = ma$
  
- $-m_1g + T_1 = -m_1a$
  
- $-m_2g + T_2 = m_2a$

- For the pulley use $\tau = I\alpha = \frac{1}{R}a$

\[ T_1R - T_2R = \frac{1}{R}a = \frac{1}{2}M\alpha \]

(Since $I = \frac{1}{2}MR^2$ for a disk)

Atwoods Machine with Massive Pulley...

- We have three equations and three unknowns ($T_1$, $T_2$, $a$). Solve for $a$.

\[ -m_1g + T_1 = -m_1a \] (1)

\[ -m_2g + T_2 = m_2a \] (2)

\[ T_1 - T_2 = \frac{1}{2}Ma \] (3)

\[ a = \frac{m_1 - m_2}{m_1 + m_2 + M}g \]