Physics 207: Lecture 20

Today's Agenda

- Homework for Monday

- Recap:
  - Systems of Particles
  - Center of mass
- Velocity and acceleration of the center of mass
- Dynamics of the center of mass

- Linear Momentum
  - Example problems
- Momentum Conservation
- Inelastic collisions in one dimension

System of Particles: Center of Mass

- For particles, the components of $R_{CM}$ are:

$$M = \sum_{i=1}^{N} m_i$$

$$(X_{CM}, Y_{CM}, Z_{CM}) = \left( \frac{\sum_i m_i x_i}{M}, \frac{\sum_i m_i y_i}{M}, \frac{\sum_i m_i z_i}{M} \right)$$

- For a continuous solid, we have to do an integral.

$$R_{CM} = \frac{\int r \, dm}{\int dm} = \frac{\int r \, dm}{M}$$

where $dm$ is an infinitesimal mass element.
System of Particles: Center of Mass

- We can use intuition to find the location of the center of mass for symmetric objects that have uniform density:
- It will simply be at the geometrical center!

\[ \text{Center of Mass} = \text{average center of mass location of the objects} \]

\[ R_{CM} = \frac{\sum_{i=1}^{N} m_i R_i}{\sum_{i=1}^{N} m_i} \]

so if we have two objects:

\[ R_{CM} = \frac{m_1 R_1 + m_2 R_2}{m_1 + m_2} \]

\[ = R_1 + \frac{m_2}{M} (R_2 - R_1) \]
Lecture 20, Act 1
Center of Mass

- The disk shown below (1) clearly has its CM at the center.
- Suppose the disk is cut in half and the pieces arranged as shown in (2):

», Where is the CM of (2) as compared to (1)?

(a) higher (b) lower (c) same

The CM of each half-disk will be closer to the fat end than to the thin end (think of where it would balance).

The CM of the compound object will be halfway between the CMs of the two halves.

This is higher than the CM of the disk.

Lecture 20, Act 1
Solution
System of Particles: Center of Mass

- The center of mass (CM) of an object is where we can freely pivot that object.

- Gravity acts on the CM of an object (show later)

- If we pivot the object somewhere else, it will orient itself so that the CM is directly below the pivot.

- This fact can be used to find the CM of oddly-shaped objects.

System of Particles: Center of Mass

- Hang the object from several pivots and see where the vertical lines through each pivot intersect!

- The intersection point must be at the CM.
Lecture 20, Act 2
Center of Mass

- An object with three prongs of equal mass is balanced on a wire (equal angles between prongs). What kind of equilibrium is this position?

a) stable
b) neutral
c) unstable

Solution

The center of mass of the object is at its center and is initially directly over the wire. If the object is pushed slightly to the left or right, its center of mass will not be above the wire and gravity will make the object fall off.
Lecture 20, Act 2
Solution

- Consider also the case in which the two lower prongs have balls of equal mass attached to them:

In this case, the center of mass of the object is below the wire. When the object is pushed slightly, gravity provides a restoring force, creating a stable equilibrium.

Velocity and Acceleration of the Center of Mass

- If its particles are moving, the CM of a system can also move.
- Suppose we know the position \( r_i \) of every particle in the system as a function of time.

\[
R_{CM} = \frac{1}{M} \sum_{i=1}^{N} m_i r_i \quad \left( M = \sum_{i=1}^{N} m_i \right)
\]

So:

\[
V_{CM} = \frac{dR_{CM}}{dt} = \frac{1}{M} \sum_{i=1}^{N} m_i \frac{dr_i}{dt} = \frac{1}{M} \sum_{i=1}^{N} m_i v_i
\]

And:

\[
A_{CM} = \frac{dV_{CM}}{dt} = \frac{1}{M} \sum_{i=1}^{N} m_i \frac{dv_i}{dt} = \frac{1}{M} \sum_{i=1}^{N} m_i a_i
\]

- The velocity and acceleration of the CM is just the weighted average velocity and acceleration of all the particles.
Linear Momentum:

- **Definition:** For a single particle, the momentum \( p \) is defined as:
  \[
  p = mv
  \]
  (\( p \) is a vector since \( v \) is a vector).

  So \( p_x = mv_x \) etc.

- Newton’s 2nd Law:
  \[
  F = ma = m \frac{dv}{dt} = \frac{d}{dt}(mv) \quad \Rightarrow \quad F = \frac{dp}{dt}
  \]

  - Units of linear momentum are \( \text{kg m/s} \).

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Linear Momentum:

- For a system of particles the total momentum \( P \) is the vector sum of the individual particle momenta:
  \[
  P = \sum_{i=1}^{N} p_i = \sum_{i=1}^{N} m_i v_i
  \]

  But we just showed that \( \sum_{i=1}^{N} m_i v_i = MV_{CM} \) \( \quad \left( V_{CM} = \frac{1}{M} \sum_{i=1}^{N} m_i v_i \right) \)

  So \( P = MV_{CM} \)
Linear Momentum:

- So the total momentum of a system of particles is just the total mass times the velocity of the center of mass.

\[ \mathbf{P} = M \mathbf{V}_{CM} \]

- Observe:

\[ \frac{d\mathbf{P}}{dt} = M \frac{d\mathbf{V}_{CM}}{dt} = M \mathbf{A}_{CM} = \sum_i m_i \mathbf{a}_i = \sum_i F_{i,\text{net}} \]

- We are interested in \( \frac{d\mathbf{P}}{dt} \) so we need to figure out \( \sum_i F_{i,\text{net}} \)

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Linear Momentum:

- Suppose we have a system of three particles as shown. Each particle interacts with every other, and in addition there is an external force pushing on particle 1.

\[ \sum_i F_{i,\text{NET}} = (F_{13} + F_{12} + F_{1,\text{EXT}}) \]

\[ + (F_{21} + F_{23}) \]

\[ + (F_{31} + F_{32}) \]

\[ = F_{1,\text{EXT}} \]

(since the other forces cancel in pairs...Newton's 3rd Law)

All of the "internal" forces cancel !!

Only the "external" force matters !!
Linear Momentum:

- Only the total external force matters!

\[ \frac{dP}{dt} = \sum_i F_{i,\text{EXT}} = F_{\text{NET,EXT}} \]

Which is the same as:

\[ F_{\text{NET,EXT}} = \frac{dP}{dt} = MA_{\text{CM}} \]

Newton’s 2nd law applied to systems!

Center of Mass Motion: Recap (really know this!)

- We have the following law for CM motion:

\[ F_{\text{EXT}} = \frac{dP}{dt} = MA_{\text{CM}} \]

- This has several interesting implications:

  - It tells us that the CM of an extended object behaves like a simple point mass under the influence of external forces:
    - We can use it to relate \( F \) and \( A \) like we are used to doing.
  - It tells us that if \( F_{\text{EXT}} = 0 \), the total momentum of the system cannot change.
    - The total momentum of a system is conserved if there are no external forces acting.
Example: Astronauts & Rope

- Two astronauts at rest in outer space are connected by a light rope. They begin to pull towards each other. Where do they meet?

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M = 1.5m
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Example: Astronauts & Rope...

- They start at rest, so $V_{CM} = 0$.
- $V_{CM}$ remains zero because there are no external forces.
- So, the CM does not move!
- They will meet at the CM.

Finding the CM:

If we take the astronaut on the left to be at $x = 0$:

$$x_{cm} = \frac{M(0) + m(L)}{M + m} = \frac{m(L)}{2.5m} = \frac{2}{5}L$$
Lecture 20, Act 3
Center of Mass Motion

- A man weighs exactly as much as his 20 foot long canoe.
- Initially he stands in the center of the motionless canoe, a distance of 20 feet from shore. Next he walks toward the shore until he gets to the end of the canoe.

What is his new distance from the shore.
(There no horizontal force on the canoe by the water).

\[
\begin{align*}
&20 \text{ ft} \\
&? \text{ ft} \\
&20 \text{ ft}
\end{align*}
\]

(a) 10 ft  
(b) 15 ft  
(c) 16.7 ft

Lecture 20, Act 3
Solution

- Since the man and the canoe have the same mass, the CM of the man-canoe system will be halfway between the CM of the man and the CM of the canoe.
- Initially the CM of the system is 20 ft from shore.

\[
\begin{align*}
&20 \text{ ft} \\
&CM \text{ of system}
\end{align*}
\]
Lecture 20, Act 3
Solution

- Since there is no force acting on the canoe in the x-direction, the location of the CM of the system can’t change!
- Therefore, the man ends up 5 ft to the left of the system CM, and the center of the canoe ends up 5 ft to the right.
- He ends up moving 5 ft toward the shore (15 ft away).

Center of Mass Motion: Review

- We have the following law for CM motion:

\[ F_{\text{EXT}} = \frac{dP}{dt} = MA_{\text{CM}} \]

- This has several interesting implications:

  - It tells us that the CM of an extended object behaves like a simple point mass under the influence of external forces:
    - We can use it to relate \( F \) and \( A \) like we are used to doing.
  - It tells us that if \( F_{\text{EXT}} = 0 \), the total momentum of the system does not change.
    - The total momentum of a system is conserved if there are no external forces acting.
Two pucks of equal mass are being pulled at different points with equal forces. Which experiences the bigger acceleration?

(a) \( A_1 > A_2 \) \hspace{1cm} (b) \( A_1 < A_2 \) \hspace{1cm} (c) \( A_1 = A_2 \)

Acceleration depends only on external force, not on where it is applied!

The answer is (c) \( A_1 = A_2 \). So the final CM velocities should be the same!
Lecture 20, Act 4
Solution

- The final velocity of the CM of each puck is the same!
- Notice, however, that the motion of the particles in each of the pucks is different (one is spinning).

\[
\begin{align*}
\text{This one has more kinetic energy (rotation)}
\end{align*}
\]

Momentum Conservation

\[
F_{\text{EXT}} = \frac{dP}{dt} \quad \Rightarrow \quad \frac{dP}{dt} = 0 \quad \Leftrightarrow \quad F_{\text{EXT}} = 0
\]

- The concept of momentum conservation is one of the most fundamental principles in physics.
- This is a component (vector) equation.
  \(\Rightarrow\) We can apply it to any direction in which there is no external force applied.
- You will see that we often have momentum conservation even when energy is not conserved.
Elastic vs. Inelastic Collisions

- A collision is said to be *elastic* when kinetic energy as well as momentum is conserved before and after the collision.
  \[ K_{\text{before}} = K_{\text{after}} \]
  Cart colliding with a spring in between, billiard balls, etc.

- A collision is said to be *inelastic* when kinetic energy is not conserved before and after the collision, but momentum is conserved.
  \[ K_{\text{before}} \neq K_{\text{after}} \]
  Car crashes, collisions where objects stick together, etc.

Inelastic collision in 1-D: Example 1

- A block of mass \( M \) is initially at rest on a frictionless horizontal surface. A bullet of mass \( m \) is fired at the block with a muzzle velocity (speed) \( v \). The bullet lodges in the block, and the block ends up with a speed \( V \). In terms of \( m, M, \) and \( V \):
  - What is the initial speed of the bullet \( v \)?
  - What is the initial energy of the system?
  - What is the final energy of the system?
  - Is kinetic energy conserved?

\[ \begin{align*}
\text{before} & \quad \quad \text{after} \\
\text{bullet} & \quad \quad \text{block + bullet}
\end{align*} \]
Example 1...

- Consider the bullet & block as a system. After the bullet is shot, there are no external forces acting on the system in the $x$-direction. **Momentum is conserved in the $x$ direction!**

$$P_{x,i} = P_{x,f}$$

$$mv = (M+m)V$$

$$v = \left(\frac{M+m}{m}\right)V$$

- Now consider the kinetic energy of the system before and after:

  - Before:
    $$E_B = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{M+m}{m}\right)^2 V^2 = \frac{1}{2}\left(\frac{M+m}{m}\right)(M+m)V^2$$

  - After: $E_A = \frac{1}{2}(M+m)V^2$

  - So $E_A = \left(\frac{m}{M+m}\right)E_B$

**Kinetic energy is NOT conserved!** (friction stopped the bullet)

However, momentum was conserved, and this was useful.
Inelastic Collision in 1-D: Example 2

Example 2...

Use conservation of momentum to find \( \mathbf{v} \) after the collision.

Before the collision: \( P_i = M \mathbf{V} + m(0) \)

After the collision: \( P_f = (M + m) \mathbf{v} \)

Conservation of momentum:

\[
\begin{align*}
P_i &= P_f \\
M \mathbf{V} &= (M + m) \mathbf{v} \\
\mathbf{v} &= \frac{M}{(M + m)} \mathbf{V}
\end{align*}
\] vector equation
\[ v = \left( \frac{M + m}{m} \right) v \]

**Example 2...**

- Now consider the K.E. of the system before and after:
  - Before:
    \[ E_{bus} = \frac{1}{2} MV^2 = \frac{1}{2} M \left( \frac{M + m}{M} \right)^2 V^2 = \frac{1}{2} \left( \frac{M + m}{M} \right) (M + m) v^2 \]
  - After:
    \[ E_A = \frac{1}{2} (M + m) v^2 \]
  - So
    \[ E_A = \left( \frac{M}{M + m} \right) E_B \]

Kinetic energy is NOT conserved in an inelastic collision!

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**Lecture 20, Act 5**

**Momentum Conservation**

- Two balls of equal mass are thrown horizontally with the same initial velocity. They hit identical stationary boxes resting on a frictionless horizontal surface.
- The ball hitting box 1 bounces back, while the ball hitting box 2 gets stuck.
  - Which box ends up moving faster?

  (a) Box 1   (b) Box 2   (c) same
Lecture 20, Act 5
Momentum Conservation

- Since the total external force in the x-direction is zero, momentum is conserved along the x-axis.
- In both cases the initial momentum is the same \((mv)\) of ball.
- In case 1 the ball has negative momentum after the collision, hence the box must have more positive momentum if the total is to be conserved.
- The speed of the box in case 1 is biggest!

\[
\text{mv}_{\text{init}} = M V_1 - m v_{\text{fin}} \\
\Rightarrow V_1 = \frac{\text{mv}_{\text{init}} + m v_{\text{fin}}}{M} \\
\text{mv}_{\text{init}} = (M+m)V_2 \\
\Rightarrow V_2 = \frac{\text{mv}_{\text{init}}}{(M+m)}
\]

\(V_1\) numerator is bigger and its denominator is smaller than that of \(V_2\).

\(V_1 > V_2\)
Explosion (inelastic un-collision)

Before the explosion:

After the explosion:

Explosion...

- No external forces, so $P$ is conserved.
- Initially: $P = 0$
- Finally: $P = m_1v_1 + m_2v_2 = 0$

$$m_1v_1 = -m_2v_2$$