Phys 207
Announcements

- Hwk 6 is posted online; submission deadline = April 4
- Exam 2 on Friday, April 8th

Today's Agenda

- Freshman Interim Grades
- Review
- Work done by variable force in 3-D
  - Newton's gravitational force
- Conservative forces & potential energy
- Conservation of “total mechanical energy”
  - Example: pendulum

Work by variable force in 3-D:

- Work $dW_r$ of a force $F$ acting through an infinitesimal displacement $\Delta r$ is:
  
  $$dW = F \cdot \Delta r$$

- The work of a big displacement through a variable force will be the integral of a set of infinitesimal displacements:
  
  $$W_{TOT} = \int F \cdot \Delta r$$
Work by variable force in 3-D:
Newton’s Gravitational Force

- Work $dW_g$ done on an object by gravity in a displacement $d\mathbf{r}$ is given by:

$$dW_g = \mathbf{F}_g \cdot d\mathbf{r} = (\frac{-GMm}{R^2}) \cdot (dR \hat{r} + Rd\theta \hat{\theta})$$

$$dW_g = (\frac{-GMm}{R^2}) dR$$ (since $\hat{r} \cdot \hat{\theta} = 0, \hat{r} \cdot \hat{r} = 1$)

Integrate $dW_g$ to find the total work done by gravity in a “big” displacement $\mathbf{R}$.

$$W_g = \int_{R_1}^{R_2} (-\frac{GMm}{R^2}) dR = GMm \left( \frac{1}{R_2} - \frac{1}{R_1} \right)$$
Work by variable force in 3-D: Newton’s Gravitational Force

- Work done depends only on \( R_1 \) and \( R_2 \), **not on the path taken.**

\[
W_g = GMm \left( \frac{1}{R_2} - \frac{1}{R_1} \right)
\]

Lecture 16, Act 1
Work & Energy

- A rock is dropped from a distance \( R_E \) above the surface of the earth, and is observed to have kinetic energy \( K_1 \) when it hits the ground. An identical rock is dropped from twice the height \((2R_E)\) above the earth’s surface and has kinetic energy \( K_2 \) when it hits. \( R_E \) is the radius of the earth.

What is \( K_2 / K_1 \)?

(a) 2
(b) \( \frac{3}{2} \)
(c) \( \frac{4}{3} \)
Lecture 16, Act 1
Solution

- Since energy is conserved, $\Delta K = W_G$.

\[ W_G = GMm \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \]
\[ \Delta K = c \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \]

Where $c = GMm$ is the same for both rocks

For the first rock:
\[ K_1 = c \left( \frac{1}{R_E} - \frac{1}{2R_E} \right) = c \cdot \frac{1}{2} \cdot \frac{1}{R_E} \]

For the second rock:
\[ K_2 = c \left( \frac{1}{R_E} - \frac{1}{3R_E} \right) = c \cdot \frac{2}{3} \cdot \frac{1}{R_E} \]

So:
\[ \frac{K_2}{K_1} = \frac{2}{3} \cdot \frac{4}{3} = \frac{8}{9} \]
Newton’s Gravitational Force Near the Earth’s Surface:

- Suppose $R_1 = R_E$ and $R_2 = R_E + \Delta y$

\[
W_g = -Gm\left(\frac{R_2 - R_1}{R_1 R_2}\right) = -Gm\left(\frac{(R_E + \Delta y) - (R_E)}{(R_E + \Delta y)(R_E)}\right) \approx -m\left(\frac{GM}{R_E^2}\right)\Delta y
\]

but we have learned that $\left(\frac{GM}{R_E^2}\right) = g$

So: $W_g = -mg\Delta y$

Conservative Forces:

- We have seen that the work done by gravity does not depend on the path taken.

\[
W_g = GMm\left(\frac{1}{R_2} - \frac{1}{R_1}\right)
\]

$W_g = -mgh$
Conservative Forces:

- In general, if the work done does not depend on the path taken (only depends the initial and final distances between objects), the force involved is said to be **conservative**.

- Gravity is a conservative force: \( W_g = GMm\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \)

- Gravity near the Earth’s surface: \( W_g = -mg\Delta y \)

- A spring produces a conservative force: \( W_s = -\frac{1}{2}k(x_2^2 - x_1^2) \)

Conservative Forces:

- We have seen that the work done by a conservative force does not depend on the path taken.

\[ W_f = W_2 \]

- Therefore the work done in a closed path is 0.

\[ W_{NET} = W_f - W_2 = W_f - W_f = 0 \]
Lecture 16, Act 2
Conservative Forces

- The pictures below show force vectors at different points in space for two forces. Which one is conservative?

(a) 1  (b) 2  (c) both

![Diagram of force vectors in x-y plane for 1 and 2]

Lecture 16, Act 2
Solution

- Consider the work done by force when moving along different paths in each case:

\[ W_A = W_B \]  \hspace{1cm}  \[ W_A > W_B \]

![Diagram showing work comparison for 1 and 2]
Lecture 16, Act 2

- In fact, you could make money on type (2) if it ever existed:
  ➡ Work done by this force in a “round trip” is \(>0\)!
  ➡ Free kinetic energy!!

\[
W_{\text{NET}} = 10 \text{ J} = \Delta K
\]

\[
W = 15 \text{ J}
\]

\[
W = 0
\]

\[
W = -5 \text{ J}
\]

\[
W = 0
\]

\[
\Delta U = U_2 - U_1 = -W = - \int_{r_1}^{r_2} F \cdot dr
\]

Note: NO REAL FORCES OF THIS TYPE EXIST, SO FAR AS WE KNOW

---

Potential Energy

- For any conservative force \(F\) we can define a potential energy function \(U\) in the following way:

\[
W = \int F \cdot dr = -\Delta U
\]

➡ The work done by a conservative force is equal and opposite to the change in the potential energy function.

- This can be written as:

\[
\Delta U = U_2 - U_1 = -W = - \int_{r_1}^{r_2} F \cdot dr
\]
Gravitational Potential Energy

- We have seen that the work done by gravity near the Earth’s surface when an object of mass \( m \) is lifted a distance \( \Delta y \) is 
  \[ W_g = -mg \Delta y \]

- The change in potential energy of this object is therefore:
  \[ \Delta U = -W_g = mg \Delta y \]

So we see that the change in \( U \) near the Earth’s surface is:

\[ \Delta U = -W_g = mg \Delta y = mg(y_2 - y_1) \]

So \( U = mg y + U_0 \) where \( U_0 \) is an arbitrary constant.

- Having an arbitrary constant \( U_0 \) is equivalent to saying that we can choose the \( y \) location where \( U = 0 \) to be anywhere we want to.
Potential Energy Recap:

- For any conservative force we can define a potential energy function $U$ such that:
  \[ \Delta U = U_2 - U_1 = -W = - \int_{S_1}^{S_2} F \cdot dr \]

- The potential energy function $U$ is always defined only up to an additive constant.

You can choose the location where $U = 0$ to be anywhere convenient.

Conservative Forces & Potential Energies (stuff you should know):

<table>
<thead>
<tr>
<th>Force $F$</th>
<th>Work $W(1-2)$</th>
<th>Change in P.E $\Delta U = U_2 - U_1$</th>
<th>P.E. function $U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_g = -mg \hat{j}$</td>
<td>$-mg(y_2-y_1)$</td>
<td>$mg(y_2-y_1)$</td>
<td>$mgy + C$</td>
</tr>
<tr>
<td>$F_y = \frac{GMM}{R^2} \hat{r}$</td>
<td>$GMM \left( \frac{1}{R_2} - \frac{1}{R_1} \right)$</td>
<td>$-GMM \left( \frac{1}{R_2} - \frac{1}{R_1} \right)$</td>
<td>$-\frac{GMM}{R} + C$</td>
</tr>
<tr>
<td>$F_s = -kx$</td>
<td>$-\frac{1}{2} k(x^2 - x^2_1)$</td>
<td>$\frac{1}{2} k(x^2_2 - x^2_1)$</td>
<td>$\frac{1}{2} kx^2 + C$</td>
</tr>
</tbody>
</table>

(R is the center-to-center distance, x is the spring stretch)
Lecture 16, Act 3
Potential Energy

- All springs and masses are identical. (Gravity acts down).

Which of the systems below has the most potential energy stored in its spring(s), relative to the relaxed position?

(a) 1
(b) 2
(c) same

Lecture 16, Act 3
Solution

- The displacement of (1) from equilibrium will be half of that of (2) (each spring exerts half of the force needed to balance mg)
Lecture 16, Act 3
Solution

The potential energy stored in (1) is \( \frac{1}{2}kd^2 = kd^2 \)

The potential energy stored in (2) is \( \frac{1}{2}k(2d)^2 = 2kd^2 \)

The spring P.E. is twice as big in (2)!

\[
\begin{align*}
E &= K + U \\
\Delta E &= \Delta K + \Delta U \\
\Delta U &= \Delta K = W \\
\Delta U &= -W = (-W) = 0
\end{align*}
\]

\[ E = K + U \text{ is constant}!! \]

Conservation of Energy

- If only conservative forces are present, the total kinetic plus potential energy of a system is conserved, i.e. the total “mechanical energy” is conserved.

\( \leftarrow \) (note: \( E=E_{\text{mechanical}} \) throughout this discussion)

- Both \( K \) and \( U \) can change, but \( E = K + U \) remains constant.
- But we’ll see that if non-conservative forces act then energy can be dissipated into other modes (thermal, sound)
Example: The simple pendulum

- Suppose we release a mass \( m \) from rest a distance \( h_1 \) above its lowest possible point.
  - What is the maximum speed of the mass and where does this happen?
  - To what height \( h_2 \) does it rise on the other side?

\[
E = \frac{1}{2}mv^2 + mgy
\]

Example: The simple pendulum

- Kinetic+potential energy is conserved since gravity is a conservative force (\( E = K + U \) is constant)
- Choose \( y = 0 \) at the bottom of the swing, and \( U = 0 \) at \( y = 0 \) (arbitrary choice)
Example: The simple pendulum

- $E = \frac{1}{2}mv^2 + mgy$.
  - Initially, $y = h_i$ and $v = 0$, so $E = mgh_i$.
  - Since $E = mgh$, initially, $E = mgh_i$ always since energy is conserved.

Example: The simple pendulum

- $\frac{1}{2}mv^2$ will be maximum at the bottom of the swing.
- So at $y = 0 \Rightarrow \frac{1}{2}mv^2 = mgh_i \Rightarrow v^2 = 2gh_i$
Example: The simple pendulum

- Since $E = mgh_1 = \frac{1}{2}mv^2 + mgy$ it is clear that the maximum height on the other side will be at $y = h_1 = h_2$ and $v = 0$.
- The ball returns to its original height.

$y = h_1 = h_2$

$y = 0$

Example: The simple pendulum

- The ball will oscillate back and forth. The limits on its height and speed are a consequence of the sharing of energy between $K$ and $U$.

$$E = \frac{1}{2}mv^2 + mgy = K + U = \text{constant.}$$
Example: The simple pendulum

- We can also solve this by choosing $y = 0$ to be at the original position of the mass, and $U = 0$ at $y = 0$.

\[ E = \frac{1}{2}mv^2 + mgy. \]

Example: The simple pendulum

- $E = \frac{1}{2}mv^2 + mgy$.
  - Initially, $y = 0$ and $v = 0$, so $E = 0$.
  - Since $E = 0$ initially, $E = 0$ always since energy is conserved.
Example: The simple pendulum

- $\frac{1}{2}mv^2$ will be maximum at the bottom of the swing.
- So at $y = -h$, $\frac{1}{2}mv^2 = mgh$, $v^2 = 2gh$.

\[ v = \sqrt{2gh} \]

Same as before!

Example: The simple pendulum

- Since $\frac{1}{2}mv^2 - mgh = 0$ it is clear that the maximum height on the other side will be at $y = 0$ and $v = 0$.
- The ball returns to its original height.

Same as before!