Lecture 1

Review of hydrogen atom

Heavy proton (put at the origin), charge $e$ and much lighter electron, charge $-e$.

Potential energy, from Coulomb's law

$$V(r) = -\frac{e^2}{4\pi \varepsilon_0} \frac{1}{r}$$

Potential is spherically symmetric. Therefore, solutions must have form

$$\Psi_{n\ell m}(r, \theta, \phi) = R_{\ell}(r) \ Y_{\ell m}(\theta, \phi)$$

$n$ is principal quantum number

$l$ is orbital angular momentum quantum number

$m$ is magnetic quantum number

Energy levels of the hydrogen atom

$$E = -\left[\frac{m_e}{2\hbar^2} \left(\frac{e^2}{4\pi \varepsilon_0}\right)^2\right] \frac{1}{\hbar^2} \rightarrow E = \frac{E_1}{\hbar^2} \ , \ n = 1, 2, 3, ...$$

**Note:** energy depends only on principal quantum number $n$ in non-relativistic approximation.

Ground state of hydrogen atom

$$n = 1 \quad e = 0 \quad m = 0$$

$$E_1 = -\left[\frac{m_e}{2\hbar^2} \left(\frac{e^2}{4\pi \varepsilon_0}\right)^2\right] = -13.6 \text{ eV}$$

Ground state wave function

$$\Psi_{100} = R_{10}(r) \ Y_{00}^{0}(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \ e^{-r/a}$$

Ground state energy
Excited states

Questions for class

1. What is the energy of the first excited state(s) (in eV)?
2. Is it degenerate?
3. If it is degenerate, how many states have the same energy and what are their quantum numbers? (ignore spin)

Answers

1. \( n=2 \)
   \[
   E_2 = \frac{E_1}{2^2} = \frac{-13.6 \text{ eV}}{4} = -3.4 \text{ eV}
   \]

2. Yes.

3. \( n = 2 \)
   \( 0 \leq l < n \) (integers)
   \( l = 0, 1 \)
   For each \( l \), \( -l \leq m \leq l \) \( \Rightarrow \)
   \( l = 0 \) \( m = 0 \) \( \{4 \text{ levels}\} \)
   \( l = 1 \) \( m = -1, 0, 1 \)

Note on spectroscopic notations (they are actually used).

There are letters associated with values of orbital angular momentum. The first few are:

- \( l = 0 \) \( s \)
- \( l = 1 \) \( p \)
- \( l = 2 \) \( d \)
- \( l = 3 \) \( f \)

For example, state with
\( n=1 \) \( l=0 \) is referred to as 1s,
\( n=2 \) \( l=0 \) is referred to as 2s,
\( n=2 \) \( l=1 \) is referred to as 2p, and so on.

While the energies are the same for the four \( n=2 \) states, the wave functions are not:

\[ R_{2s} = \frac{1}{\sqrt{12}} a^{-3/2} \left( 1 - \frac{r}{a} \right) \exp\left(-\frac{r}{2a}\right) \]
\[ R_{2p} = \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} \exp\left(-\frac{r}{2a}\right) \]
\[ \psi_{210} = R_{2s} Y_1^0 \]
\[ \psi_{211} = R_{2s} Y_1^1 \]
\[ \psi_{21-1} = R_{2s} Y_1^{-1} \]
\[ \psi_{211} = R_{2s} Y_1^1 \]

\[ Y_0^0 = \frac{1}{\sqrt{4\pi}} \]
\[ Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta \]
\[ Y_{1}^{\pm 1} = \frac{3}{\sqrt{8\pi}} \sin \theta e^{\pm i\phi} \]
Angular momentum: summary

\[
\begin{align*}
\text{If} & \quad [ L_x, L_y ] = i \hbar L_z \\
& \quad [ L_y, L_z ] = i \hbar L_x \\
& \quad [ L_z, L_x ] = i \hbar L_y \quad \text{then}
\end{align*}
\]

Eigenfunctions \( f^m_l \) of \( L^2 \) and \( L_z \) are labeled by \( m \) and \( l \):

\[
\begin{align*}
\text{eigenvalue of } L^2 & = \hbar^2 l(l+1) f^m_l, \\
\text{eigenvalue of } L_z & = \hbar m f^m_l.
\end{align*}
\]

\( l = 0, 1, 2, \ldots \) (only integer values for orbital angular momentum)

For a given value of \( l \), there are \( 2l+1 \) values of \( m \): \( m = -l, -l+1, \ldots, l-1, l \).

Generally, half-integer values of the angular momentum are also allowed but not for orbital angular moment.

\[
L_z = -i \hbar \frac{\partial}{\partial \phi}
\]

\[
L^2 = -\hbar^2 \left[ \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left( \sin \Theta \frac{\partial}{\partial \Theta} \right) + \frac{1}{\sin^2 \Theta} \frac{\partial^2}{\partial \phi^2} \right].
\]

Elementary particles carry intrinsic angular momentum \( S \) in addition to \( L \). Spin of elementary particles has nothing to do with rotation, does not depend on coordinates \( \Theta \) and \( \Phi \), and is purely a quantum mechanical phenomena.
Spin $\frac{1}{2}$

$s = \frac{1}{2}$, therefore $m = \pm \frac{1}{2}$ and there are two eigenstates $|s m\rangle = |\frac{1}{2} \frac{1}{2}\rangle$, $|s m\rangle = \frac{1}{\sqrt{2}} \left( \frac{1}{2} \pm \frac{i}{2} \right) |\frac{1}{2} \frac{1}{2}\rangle$. We will call them spin up $\uparrow |\frac{1}{2} \frac{1}{2}\rangle$ and spin down $\downarrow |\frac{1}{2} -\frac{1}{2}\rangle$.

Taking these eigenstates to be basis vectors, we can express any spin state of a particle with spin $\frac{1}{2}$ as:

$$\gamma = \begin{pmatrix} a \\ b \end{pmatrix} = a \gamma_+ + b \gamma_-$$

represents spin up $\uparrow$

represents spin down $\downarrow$

$\gamma_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\gamma_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

All our spin operators are 2x2 matrixes for spin $\frac{1}{2}$, which we can find out from how they act on our basis set states $\gamma_+$ and $\gamma_-$. Raising and lowering operators:

$$S_\pm = S_x \pm i S_y$$

$$S_\pm |s m\rangle = \frac{\hbar}{2} \sqrt{s(s+1) - m(m\pm 1)} \langle s(m\pm 1)|$$
**Class exercise**

The electron in a hydrogen atom occupies the combined spin an position state:

\[
\Psi = R_{2,1} \left( \sqrt{\frac{1}{3}} Y_{1}^{0} \gamma_{+} + \sqrt{\frac{2}{3}} Y_{1}^{1} \gamma_{-} \right)
\]

\[n=2 \quad \ell=1 \quad s=\frac{1}{2} \quad m_{\ell}=0 \quad m_{s}=\frac{1}{2} \quad \ell=1 \quad m_{\ell}=1 \quad m_{s}=-\frac{1}{2}\]

Note that \( m_{\ell} + m_{s} = \frac{1}{2} \) in both cases

(a) If you measure the orbital angular momentum squared \( L^2 \), what values might you get and what is the probability of each?

\[L^2 \Psi = \ell(\ell+1) \hbar^2 \Psi \Rightarrow \]

You get \( \hbar^2 \ell(\ell+1) = 2 \hbar^2 \) with probability \( P=1 \) (100%).

(b) Same for z component of the orbital angular momentum \( L_z \).

\[L_z \Psi = \hbar m_e \Psi \]

Possible values of \( m_e \): \( m_e = 0 \) or \( 1 \).

\[P = \left( \frac{1}{3} \right)^2 = \frac{1}{3} \quad \text{for} \quad m_e = 0\]

\[P = \left( \frac{2}{3} \right)^2 = \frac{4}{3} \quad \text{for} \quad m_e = 1\]

(c) Same for the spin angular momentum squared \( S^2 \).

\[S^2 \Psi = \hbar^2 s(s+1) \Psi \quad s=\frac{1}{2} \Rightarrow \text{You get } \frac{3}{4} \hbar^2 \text{ with } P=1.\]

(d) Same for z component of the spin angular momentum \( S_z \).

\[S_z \Psi = \hbar m_s \Psi \quad m_s = \frac{1}{2} \text{ and } -\frac{1}{2}\]

\[P = \frac{1}{3} \text{ for } m_s = \frac{1}{2} \quad P = \frac{2}{3} \text{ for } m_s = -\frac{1}{2}\]
Addition of angular momenta

Let's go back to ground state of hydrogen: it has one proton with spin $\frac{1}{2}$ and one electron with spin $\frac{1}{2}$ (orbital angular momentum is zero). What is the total angular momentum $\vec{S}$ of the hydrogen atom?

\[ \vec{S} = \vec{S}_1 + \vec{S}_2 \]

Total spin

Electron's spin, acts only on electron's spin states

Proton's spin, acts only on proton's spin states

\[ S_z \chi_1 \chi_2 = \left( S_z^{(1)} + S_z^{(2)} \right) \chi_1 \chi_2 = \]

\[ = \left( S_z^{(1)} \chi_1 \right) \chi_2 + \chi_1 \left( S_z^{(2)} \chi_2 \right) = \hbar m_1 \chi_1 \chi_2 + \hbar m_2 \chi_1 \chi_2 \]

\[ = \hbar \left( m_1 + m_2 \right) \chi_1 \chi_2 \]

Therefore, the z components just add together and quantum number $m$ for the composite system is simply

\[ m = m_1 + m_2. \]

There are four possible combinations:

- $m_1 = \frac{1}{2}, \ m_2 = \frac{1}{2}$ \quad $\uparrow \uparrow \quad m = 1$
- $m_1 = \frac{1}{2}, \ m_2 = -\frac{1}{2}$ \quad $\uparrow \downarrow \quad m = 0$
- $m_1 = -\frac{1}{2}, \ m_2 = -\frac{1}{2}$ \quad $\downarrow \uparrow \quad m = 0$
- $m_1 = -\frac{1}{2}, \ m_2 = \frac{1}{2}$ \quad $\downarrow \downarrow \quad m = -1$

(first arrow corresponds to the electron spin and second arrow corresponds to the nuclear spin).
Well, it appears that we have an extra state!

Let's apply lowering operator to state $\uparrow \uparrow$ to sort this out:

$$S_\downarrow |s m\rangle = \frac{1}{\sqrt{2}} \sqrt{s(s+1) - m(m-1)} |s m-1\rangle$$

$$S_-^{(1)} \uparrow = S_-^{(1)} \frac{1}{2} \frac{1}{2} \uparrow = \frac{1}{\sqrt{2}} \sqrt{\frac{3}{2} - \frac{1}{2} (\frac{1}{2} - 1)} \frac{1}{2} \frac{1}{2} \uparrow = \frac{1}{\sqrt{2}} \downarrow$$

state with $s = \frac{1}{2}, m = \frac{1}{2}$

$$S_-^{(2)} \uparrow = S_-^{(2)} \frac{1}{2} \frac{1}{2} \uparrow = \frac{1}{\sqrt{2}} \sqrt{\frac{3}{2} - \frac{1}{2} (\frac{1}{2} - 1)} \frac{1}{2} \frac{1}{2} \uparrow = \frac{1}{\sqrt{2}} \downarrow$$

state with $s = \frac{1}{2}, m = -\frac{1}{2}$

$$S_- (\uparrow \uparrow) = (S_-^{(1)} + S_-^{(2)}) \uparrow \uparrow$$

$$= (S_-^{(1)} \uparrow) \uparrow + \uparrow (S_-^{(2)} \uparrow) = \frac{1}{\sqrt{2}} \downarrow \uparrow + \frac{1}{\sqrt{2}} \uparrow \downarrow = \frac{1}{\sqrt{2}} (\downarrow \uparrow + \uparrow \downarrow)$$

(Note: normalization is not preserved here).

So we can sort out four states as follows:

Three states $|s m\rangle$ with spin $s = 1$, $m = 1, 0, -1$:

$$\begin{cases} 
|11\uparrow\rangle = \uparrow \uparrow \\
|10\downarrow\rangle = \frac{1}{\sqrt{2}} (\uparrow \downarrow + \downarrow \uparrow) \\
|1-1\downarrow\rangle = \downarrow \downarrow 
\end{cases} \quad S = 1$$

This is called a \textbf{triplet} configuration.

and one state with spin $s = 0$, $m = 0$:

$$\begin{cases} 
|00\rangle = \frac{1}{\sqrt{2}} (\uparrow \downarrow - \downarrow \uparrow) 
\end{cases} \quad S = 0$$

This is called a \textbf{singlet} configuration.

\textbf{Summary: Combination of two spin $\frac{1}{2}$ particles can carry a total spin of $s = 1$ or $s = 0$, depending on whether they occupy the triplet or singlet configuration.}
**Addition of angular momenta**

Ground state of hydrogen: it has one proton with spin $\frac{1}{2}$ and one electron with spin $\frac{1}{2}$ (orbital angular momentum is zero). What is the total angular momentum $\vec{S}$ of the hydrogen atom?

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

Total spin

Electron's spin, acts only on electron's spin states

Proton's spin, acts only on proton's spin states

The $z$ components just add together and quantum number $m$ for the composite system is simply

$$m = m_1 + m_2$$

In general, it you combine any angular momentum $\vec{j}_1$ and $\vec{j}_2$ you get every value of angular momentum from $|\vec{j}_1 - \vec{j}_2|$ to $\vec{j}_1 + \vec{j}_2$ in integer steps:

$$\vec{j} = |\vec{j}_1 - \vec{j}_2|, \ldots, (\vec{j}_1 + \vec{j}_2)$$

It does not matter if it is orbital angular momentum or spin.

**Example:**

$$\vec{j}_1 = \frac{3}{2}, \quad \vec{j}_2 = 3 \quad \Rightarrow$$

$$|\vec{j}_1 - \vec{j}_2| = \left| \frac{3}{2} - 3 \right| = \frac{3}{2}$$

$$\vec{j}_1 + \vec{j}_2 = \frac{3}{2} + 3 = \frac{9}{2}$$

Total angular momentum $\vec{j}$ can be $\frac{3}{2}, \frac{5}{2}, \frac{7}{2},$ and $\frac{9}{2}$. 
Problem

Quarks carry spin \( \frac{1}{2} \). Two quarks (or actually a quark and an antiquark) bind together to make a meson (such as pion or kaon). Three quarks bind together to make a barion (such as proton or neutron). Assume all quarks are in the ground state so the orbital angular momentum is zero.

(1) What spins are possible for mesons?

(2) What spins are possible for baryons?

Solution:

(1) \( S_1 = \frac{1}{2} \quad S_2 = \frac{1}{2} \implies S = 0 \) or 1.

(2) \( S_1 = \frac{1}{2} \quad S_2 = \frac{1}{2} \quad S_3 = \frac{1}{2} \)

Add these first

\( S_{12} = 0 \) or 1

Now add the third spin:

\( S_{12} = 0 \quad S_3 = \frac{1}{2} \implies S = \frac{1}{2} \)

\( S_{12} = 1 \quad S_3 = \frac{1}{2} \implies S = \left| \frac{1}{2} - 1 \right| \quad \left( \frac{1}{2} + 1 \right) = \frac{3}{2} \)

\( S = \frac{1}{2} \) or \( \frac{3}{2} \)