Lecture 11  Problem solving

Problem 1
A particle of mass m in the harmonic oscillator potential starts out in the state

\[ \Psi(x, 0) = A \left( 1 - 2 \sqrt{\frac{m \omega}{\hbar}} x \right)^2 e^{-\frac{m \omega}{2 \hbar} x^2} \]

for some constant A.

(a) What is the expectation value of the energy?

(b) At some later time T the wave function is

\[ \Psi(x, T) = B \left( 1 \UPARROW \bigcirc \bigcirc 2 \sqrt{\frac{m \omega}{\hbar}} x \right)^2 e^{-\frac{m \omega}{2 \hbar} x^2} \]

for some constant B. What is the smallest possible value of T?

Solution
First, let's introduce standard notations for harmonic oscillator:

\[ \xi = \sqrt{\frac{m \omega}{\hbar}} x \quad \lambda = \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \]

Then, \( \Psi(x, 0) = A \left( 1 - 2 \xi \right)^2 e^{-\xi^2/2} \]

\[ = A \left( 1 - 4 \xi + 4 \xi^2 \right) e^{-\xi^2/2} \]

This function can be expressed as a linear combination of the first three states of harmonic oscillator.

\[ \Psi(x, 0) = c_0 \psi_0(x) + c_1 \psi_1(x) + c_2 \psi_2(x) \]
Now, we need to find coefficients \( c \) by equating same powers of \( \xi \):

\[
\begin{align*}
\psi_0(x) &= c e^{-\xi^2/2} \\
\psi_1(x) &= \sqrt{2} \psi \xi e^{-\xi^2/2} \\
\psi_2(x) &= \frac{d}{\sqrt{2}} \left( 2 \xi^2 - 1 \right) e^{-\xi^2/2}
\end{align*}
\]

Normalization gives:

\[
\psi(x, 0) = (A - 4A \xi + 4A \xi^2) e^{-\xi^2/2}
\]

\[
\psi(x, 0) = (d \psi_0 + \sqrt{2} \psi_2) x \xi + \frac{d}{\sqrt{2}} \cdot 2 \psi_2 \xi^2 - \frac{d}{\sqrt{2}} \psi_2
\]

\[
\xi_0: \quad A = d \psi_0 - \frac{d}{\sqrt{2}} \psi_2 = d \psi_0 - \frac{d}{\sqrt{2}} \cdot 2 \sqrt{2} A \xi = d \psi_0 - 2A \quad \text{c.o. } 3A/d
\]

\[
\xi_1: \quad -4A = \sqrt{2} \psi_2 \quad \Rightarrow \quad c_1 = -2 \sqrt{2} A / d
\]

\[
\xi_2: \quad 4A = d \psi_2 \quad \Rightarrow \quad c_2 = 2 \sqrt{2} A / d
\]

Normalization gives:

\[
A = |c_0|^2 + |c_1|^2 + |c_2|^2
\]

\[
= 9 \frac{A^2}{\alpha^2} + 8 \frac{A^2}{\alpha^2} + 8 \frac{A^2}{\alpha^2} = 25 \frac{A^2}{\alpha^2} \quad \Rightarrow \quad A = \frac{d}{5}
\]

\[
c_0 = \frac{3}{5}, \quad c_1 = -\frac{2 \sqrt{2}}{5}, \quad c_2 = \frac{2 \sqrt{2}}{5}
\]
Now it is really easy to find the expectation value of energy:

\[
\langle H \rangle = \sum_n |c_n|^2 E_n = \sum_n |c_n|^2 \left(n + \frac{1}{2}\right) \hbar \omega
\]

**Proof:**

\[
H \psi_n = E_n \psi_n
\]

\[
\langle H \rangle = \int \psi^* H \psi \, dx = \int \left(\sum_{m} c_m \psi_m^* \right) H \left(\sum_{n} c_n \psi_n \right) \, dx
\]

\[
= \sum_{m} \sum_{n} c_m^* c_n E_n \int \psi_m^* \psi_n \, dx = \sum_{n} |c_n|^2 E_n
\]

\[
\langle H \rangle = C_0^2 \left(\frac{1}{2} \hbar \omega\right) + C_1^2 \left(\frac{3}{2} \hbar \omega\right) + C_2^2 \left(\frac{5}{2} \hbar \omega\right)
\]

\[
= \frac{9}{25} \left(\frac{1}{2} \hbar \omega\right) + \left(\frac{8}{25}\right) \left(\frac{3}{2} \hbar \omega\right) + \left(\frac{8}{25}\right) \left(\frac{3}{2} \hbar \omega\right)
\]

\[
\langle H \rangle = \frac{23}{50} \hbar \omega
\]

(b) \(\psi(x, t) = \sum_{n=0}^{2} c_n \psi_n(x) e^{-iE_n t/\hbar}\) using \(E = \hbar \omega\)

we get

\[
\psi(x, t) = \frac{3}{5} \psi_0 e^{-i\omega t/2} - \frac{2\sqrt{2}}{5} \psi_1 e^{-3i\omega t/2} + \frac{2\sqrt{2}}{5} \psi_2 e^{-5i\omega t/2}
\]

\[
= e^{-i\omega t/2} \left[ \frac{3}{5} \psi_0 - \frac{2\sqrt{2}}{5} e^{-i\omega t} \psi_1 + \frac{2\sqrt{2}}{5} \psi_2 e^{-2i\omega t} \right]
\]

to change sign of this term \(e^{-i\omega T} = 1 \Rightarrow\)

\(\omega T = \pi\) and \(T = \pi/\omega\)

Note that \(e^{-2i\omega T} = 1\) and sign of the last term does not change.
Problem 2

(a) Show that the wave function of a particle in the infinite square well returns to its original form after a quantum revival time \( T = \frac{4m a^2}{\pi^2 \hbar} \), i.e.
\[
\psi(x, T) = \psi(x, 0)
\]
for any state (not just a stationary state).

(b) What is the classical revival time, for a particle of energy \( E \) bouncing back and forth between the walls?

(c) For what energy are the two revival times equal?

Solution

The most general solution for the infinite square well potential is:
\[
\psi(x, t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-i \left( \frac{n^2 \hbar^2 \pi^2}{2ma^2} \right) t}
\]

Therefore,
\[
\frac{n^2 \hbar^2 \pi^2}{2ma^2} T = \frac{n^2 \hbar^2 \pi^2}{2ma^2} \left( \frac{4ma^2}{\pi^2 \hbar} \right) = 2\pi n^2
\]

plug in revival time

\[
-e^{-i \left( \frac{n^2 \hbar^2 \pi^2}{2ma^2} \right) (t+T)} = e^{-i \left( \frac{n^2 \hbar^2 \pi^2}{2ma^2} \right) t} e^{i 2\pi n^2} = 1 \text{ since } n^2 \text{ is an integer}
\]

We get
\[
\psi(x, t+T) = \psi(x, t)
\]
(b) The classical revival time is the time that particle travels from one side of the well to the other and back.

\[ 2a = vT \implies T_c = \frac{2a}{v} \]

Since \( V = 0 \) in the well \( \implies E = E_{kin} = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2E}{m}} \) and \( T_c = \frac{2a}{\sqrt{2E}} \sqrt{m} = a\sqrt{\frac{2m}{E}} \)

(c) Quantum and classical revival times are equal if

\[ T_q = \frac{4ma^2}{\pi^2\hbar^2} = a\sqrt{\frac{2m}{E}} \Rightarrow \]

\[ \frac{16m^2a^2}{\pi^2\hbar^2} = \frac{2m}{E} \Rightarrow \]

\[ E = \frac{\pi^2\hbar^2}{8ma^2} = \frac{E_1}{4} \]
Problem 3

Let \( P_{ab}(t) \) be the probability of finding a particle in the range \( a<x<b \), at time \( t \).

(a) Show that

\[
\frac{dP_{ab}}{dt} = J(a, t) - J(b, t)
\]

where

\[
J(x, t) = \frac{i\hbar}{2m} \left( \psi \frac{d\psi^*}{dt} - \psi^* \frac{d\psi}{dt} \right).
\]

The \( J(x, t) \) is called probability current since it tells you the rate with which probability is "flowing" past the point \( x \). What are its units?

(b) Find the probability current for the wave function

\[
\psi(x, t) = A e^{-\frac{a}{2} \left( \frac{m x^2}{\hbar^2} + it \right)}.
\]

Solution

\[
P_{ab}(t) = \int_a^b |\psi(x, t)|^2 \, dx
\]

\[
\frac{dP_{ab}}{dt} = \int_a^b \frac{2}{\hbar} |\psi|^2 \, dx
\]

In one of the lectures, we found that

\[
\frac{2}{\hbar} |\psi|^2 = \frac{2}{\hbar} \left[ \frac{i\hbar}{2m} \left( \psi^* \frac{d\psi}{dx} - \frac{d\psi^*}{dx} \psi \right) \right]
\]

Comparing it with definition of \( J(x, t) \), we get
\[ \frac{\partial |\psi|^2}{\partial t} = - \frac{\partial}{\partial x} \mathcal{J}(x, t) \]

\[ \frac{d \mathcal{P}_{ab}}{dt} = - \int_a^b \frac{2}{\partial x} \mathcal{J}(x, t) \, dx = - \left[ \mathcal{J}(x, t) \right]_a^b \]

\[ = \mathcal{J}(a, t) - \mathcal{J}(b, t). \quad \text{QED} \]

Probability is dimensionless, so \( \mathcal{J} \) has dimensions 1/time, and units \((\text{seconds})^{-1}\).

\( (b) \quad \psi(x, t) = A e^{-a \left[ \frac{m}{\hbar} x^2 / t + iat \right]} \)

\[ = f(x) e^{-iat} \]

\[ f(x) = A e^{-amx^2 / \hbar} \]

\[ \frac{\partial \psi}{\partial x} = f(x) e^{-iat} \frac{df}{dx} e^{iat} = f \frac{df}{dx} \]

\[ \psi^* \frac{\partial \psi}{\partial x} = \left( f(x) e^{iat} \frac{df}{dx} e^{-iat} \right) = f \frac{df}{dx} \]

\[ \mathcal{J}(x, t) = 0 \]
Note on the calculation of integral \( \int_{-\infty}^{\infty} e^{-\left(a x^2 + bx\right)} \, dx \) in Homework #4.

Change of variables

\[
y = \sqrt{a} \left( x + \frac{b}{2a} \right)
\]

\[
x = \frac{y}{\sqrt{a}} - \frac{b}{2a}
\]

\[
a x^2 + bx = a \left( \frac{y}{\sqrt{a}} - \frac{b}{2a} \right)^2 + b \left( \frac{y}{\sqrt{a}} - \frac{b}{2a} \right)
\]

\[
= a \left( \frac{y^2}{a} - \frac{2yb}{2a\sqrt{a}} + \frac{b^2}{4a^2} \right) + b \frac{y}{\sqrt{a}} - \frac{b^2}{2a}
\]

\[
= y^2 - \frac{b^2}{4a}
\]

Therefore,

\[
\int_{-\infty}^{\infty} e^{-\left(a x^2 + bx\right)} \, dx = \int_{-\infty}^{\infty} e^{-y^2 + \frac{b^2}{4a}} \frac{1}{\sqrt{a}} \, dy
\]

\[
= \frac{1}{\sqrt{a}} e^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} e^{-y^2} \, dy = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}
\]

\[
\int_{-\infty}^{\infty} e^{-\left(a x^2 + bx\right)} \, dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}
\]
We used

\[ \int_{-\infty}^{\infty} x^{2n} e^{-x^2/a^2} \, dx = \sqrt{\pi} \frac{(2n)!}{n!} \left( \frac{a}{2} \right)^{2n+1} \]

for \( n = 1 \) and \( a = 1 \), we find

\[ \int_{-\infty}^{\infty} e^{-y^2} \, dy = 2 \sqrt{\pi} \frac{1}{2} = \sqrt{\pi} \]