Lecture 13

The finite square well

\[ V(x) = \begin{cases} -V_0 & -a < x < a \\ 0 & |x| > a \end{cases} \]

Bound states: \( E < 0 \)
Scattering states: \( E > 0 \)

**Bound states**

**Step 1: Solve Schrödinger equation for all regions**

1. **Region** \( x < -a \), \( V = 0 \) and \( -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi \) (\( E < 0 \))

\[ \frac{d^2\psi}{dx^2} = k^2\psi \] where \( k = \sqrt{2mE/\hbar} \) is real and positive

\[ \psi = A e^{-kx} + B e^{kx} \Rightarrow \psi(x) = B e^{kx} \]

\( x < -a \)

2. **Region** \(-a < x < a\), \( V(x) = -V_0 \)

\[ -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} - V_0 \psi = E\psi \]

\( \ell = \frac{\sqrt{2m(E + V_0)}}{\hbar} \) is real and positive (\( E > V_{\text{min}} \))

General solution is \( \psi = c_1 e^{i\ell x} + c_2 e^{-i\ell x} \) but we will use \( \sin \ell x \) and \( \cos \ell x \) form to distinguish even and odd solutions.

\[ \psi(x) = C \sin (\ell x) + D \cos (\ell x) \quad \text{for} \ -a < x < a \]
Step 2: Apply boundary conditions that $\psi$ and $d\psi/dx$ are continuous at $a$ and $-a$.

Note: $V(x)$ is even $\Rightarrow$ wave functions are either even or odd. Therefore, we only need to impose conditions on one side, and use $\psi(-x) = \pm \psi(x)$ for the other side.

Even solutions:

$$\psi(x) = \begin{cases} F e^{-kx} & x > a \\ D \cos(kx) & 0 < x < a \\ \psi(-x) & x < 0 \end{cases}$$

(1) $\psi$ is continuous at $a$ $\Rightarrow$

$$F e^{-ka} = D \cos(ka) \quad (1)$$

(2) $d\psi/dx$ is continuous at $a$

$$-Fk e^{-ka} = -D \sin(ka) \quad (2)$$

We divide equations (2) and (1) to get

$$\frac{F e^{-ka}}{e^{-ka}} = \frac{D \sin(ka)}{D \cos(ka)}$$

$$k = \ell \tan(\ell a)$$
This equation gives you formula for allowed energies (remember that \( k = \frac{\sqrt{-2mE}}{\hbar} \) and \( \ell = \sqrt{2m(E + V_0)} \)).

One can solve this equation numerically. First, we change variables:

\[
\zeta \equiv \ell a; \quad \zeta_0 = \frac{a}{\frac{\hbar}{k}} \sqrt{2mV_0}
\]

\[\kappa = \ell \tan(\ell \zeta) \Rightarrow \tan \zeta = \sqrt{(\zeta_0/\ell)^2 - 1}
\]

Check:
\[
\frac{\ell}{\kappa} = \sqrt{\frac{\zeta_0^2}{\ell^2} - 1} = \frac{1}{\ell a} \sqrt{\frac{\zeta_0^2}{\ell^2} - \zeta^2}
\]

\[\Rightarrow \kappa a = \sqrt{\zeta_0^2 - \zeta^2} = \sqrt{\frac{a^2}{\kappa^2} 2mV_0 - 2m(V_0 + E) \frac{a^2}{\hbar^2}} = \frac{a}{\frac{\hbar}{k}} \sqrt{-2mE} = \kappa a \text{ (OK)}
\]

**Solutions of equation**

\[\tan \zeta = \sqrt{(\zeta_0/\ell)^2 - 1}
\]
Step 3. Normalize $\Psi$ (Find D and F). (Homework).

Scattering states

$E > 0$
Again, step 1: solve Schrödinger equation for all regions.

$\chi < -a : V = 0 : \Psi(x) = Ae^{i\kappa x} + Be^{-i\kappa x}$
$\kappa = \frac{\sqrt{2mE}}{\hbar}$

$-a < x < a : V = -V_0 : \Psi(x) = C\sin(\lambda x) + D\cos(\lambda x)$
$\lambda = \sqrt{\frac{2m(V_0 + E)}{\hbar^2}}$

$x > a$ (assuming no incoming wave from the right)
$\Psi(x) = Fe^{i\kappa x}$
($C = 0$)

A is amplitude of incoming wave (from the left).
B is the reflected amplitude and F is the transmitted amplitude.
Since the scattering problem is asymmetric, we use exp notations instead of sin/cos ones.

Step 2. Apply boundary conditions.

There are four boundary conditions:

1. $\Psi$ is continuous at $a$.
2. $\Psi$ is continuous at $-a$.
3. $\frac{d\Psi}{dx}$ is continuous at $a$.
4. $\frac{d\Psi}{dx}$ is continuous at $-a$.
\(4(a)\) (1) gives: \(C \sin(\lambda a) + D \cos(\lambda a) = Fe^{i\kappa a}\)

\(4(-a)\) (2) gives: \(A e^{-i\kappa a} + B e^{i\kappa a} = -C \sin(\lambda a) + D \cos(\lambda a)\)

\(\frac{dy}{dx}\bigg|_{a}\) (3) gives: \(i \hbar F e^{i\kappa a}\)

\(\frac{dy}{dx}\bigg|_{-a}\) (4) gives: \(i \hbar \left[A e^{-i\kappa a} - B e^{i\kappa a}\right] = \hbar [C \cos(\lambda a) - D \sin(\lambda a)]\)

We can use (1) and (3) to eliminate \(C\) and \(D\) and solve remaining for \(B\) and \(F\):

\[
B = i \frac{\sin(2\lambda a)}{2\hbar} \left( e^2 - k^2 \right) F
\]

\[
F = \frac{e^{-2i\kappa a}}{\cos(2\lambda a) - i \frac{(k^2 + \lambda^2)}{2\hbar} \sin(2\lambda a)} A
\]

Transmission coefficient \(T = \frac{|F|^2}{|A|^2}\)
Exercise 6 (homework problem)

Consider "step" potential

\[ V(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
V_0 & \text{if } x > 0
\end{cases} \]

(a) Calculate the reflection coefficient, for \( E < V_0 \) and comment on the result.

(b) Calculate the reflection coefficient for the case \( E > V_0 \).