Chapter 20 – Examples

1. A particular electric circuit consists of a resistor (denote as \( R_A \)), and a light bulb connected in series to an AC source (wall outlet) that provides of \( V_{rms} = 120 \text{V} \). In a circuit without the resistor, the bulb uses 100 W of power.

   a) Draw a circuit for the light bulb and resistor \( A \) in series with the AC source. Determine the value of the resistance needed for resistor \( A \) if the light bulb is to use 75.0 W of power.

   First, find the resistance of the bulb alone

   \[
   P = V_{rms}I_{rms} \quad \Rightarrow \quad I = \frac{V}{R}
   \]

   \[
   P = \frac{V^2}{R_b}
   \]

   \[
   R_b = \frac{V^2}{P} = \frac{(120 \text{V})^2}{100 \text{W}} = 144\Omega
   \]

   Since the resistor and bulb are in series, the current must be the same through both and the voltage drop across both is the sum of the voltage drops across each:

   \[
   I = I_b = I_A \quad \quad V = V_b + V_A
   \]

   The net resistance in series is the sum of the resistances, \( R = R_b + R_A \)

   The current can be found through Ohm’s law.

   \[
   V = I(R_b + R_A) \quad \quad V_b = IR_b
   \]

   \[
   I = \frac{V}{R_b} = \frac{V}{R_b + R_A}
   \]

   \[
   V_b = \frac{VR_b}{R_b + R_A}
   \]

   The power through the bulb is

   \[
   P_b = V_bI \quad \quad P_b = \frac{V_b^2}{R_b}
   \]

   \[
   P_b = \left( \frac{VR_b}{R_b + R_A} \right)^2 = \frac{V^2R_b}{(R_b + R_A)^2}
   \]

   Solve for the value of resistor \( A \):
(R_b + R_A)^2 = \frac{V^2R_b}{P_b}

R_A = \sqrt{\frac{V^2R_b}{P_b} - R_b} = \sqrt{\frac{(120 \text{ V})^2 (144 \Omega)}{75 \text{ W}}} - 144 \Omega

R_A = 22.3 \Omega

b) Determine the power output of the AC source and compare to the case without the resistor.

With resistor:

\[ P = VI = \frac{V^2}{R} \]

\[ P = \frac{V^2}{R_b + R_A} = \frac{V^2}{R_b + \left(\sqrt{\frac{V^2R_b}{P_b} - R_b}\right)} \]

\[ P = \frac{V^2}{\sqrt{\frac{V^2R_b}{P_b}}} = V\sqrt{\frac{P_b}{R_b}} \]

\[ P = \frac{V^2}{\sqrt{\frac{V^2R_b}{P_b}}} = V\sqrt{\frac{P_b}{R_b}} \]

\[ P = (120 \text{ V})\sqrt{\frac{75 \text{ W}}{144 \Omega}} = 86.6 \text{ W} \]

In the case the power reduction is not 1/4, but is the square root of 1/4. If the output of the bulb was reduced to x of the original value by adding a resistor in series, the battery power output is only decreased to the square root of x.

c) A second identical light bulb replaces resistor A. They are placed in series with the AC generator (i.e. wall socket), then in parallel with the AC generator. The AC generator provides \( V_{rms} = 120 \text{V} \). Calculate the average power delivered by the AC generator in each case to find the ratio of light emitted of both cases.

Series case:

Since the bulbs are in series, the current must be the same through both and the voltage drop across both is the sum of the voltage drops across each:

\[ I = I_1 = I_2 \quad \text{and} \quad V = V_1 + V_2 \]

The net resistance in series is the sum of the resistances, \( R = R_b + R_b \)
\[ P = VI \quad P = \frac{V^2}{R} \]

The power delivered by the AC source is \( P = \frac{V^2}{2R_b} \).

Parallel Case:
Here, the bulbs are in parallel, the current must be the sum of the currents across each and the voltage drop across both is the same as the voltage drops across each:

\[ I = I_b + I_A \quad V = V_b = V_A \]

The net resistance in parallel is \( \frac{1}{R} = \frac{1}{R_b} + \frac{1}{R_b} \Rightarrow R = \frac{R_b}{2} \)

\[ P = VI \quad P = \frac{V^2}{R} \]

The power delivered by the AC source is \( P = \frac{2V^2}{R_b} \).

It is easy to see that the power delivered in the parallel case is four times more than the serial case. Since the only items using power is the light bulbs. This is also the ratio of light emitted.

2. In “Back to the Future”, the time machine is said to need 1.21 gigawatts of power. Doc Brown runs a cable to the top of the town hall to make a connection to a wire hanging over the street thus providing a connection to the DeLorean as it speeds past. Assume the entire length (from the lightning rod to the car) is a 100 m iron cable of resistivity \( \rho = 9.7 \times 10^{-8} \Omega \cdot \text{m} \).

a) The typical lightning bolt has a current of about 40.0 kA. What is the voltage needed to transfer 1.21 gigawatts of power?

\[ P = VI \quad \Rightarrow \quad V = \frac{P}{I} \]

\[ V = \frac{1.21 \times 10^9 \text{ W}}{4.00 \times 10^4 \text{ A}} = 3.03 \times 10^4 \text{ V} \]

b) If the cable is 1.5 cm in diameter, how much power is lost with 40.0 kA running through the cable?
\[ P = VI \quad \Rightarrow \quad P = I^2 R \]
\[ R = \rho \frac{L}{A} \]
\[ P = I^2 \rho \frac{L}{A} = \left(4.0 \times 10^4 \text{ A}\right)^2 \left(9.61 \times 10^{-8} \Omega \cdot \text{m}\right) \frac{100 \text{ m}}{(0.015 \text{ m})^2} \]
\[ P = 6.83 \times 10^7 \text{ W} \]

Or about 68 MW.

c) Assume the resistance in part (b) was calculated at 20°C. What would the power loss through the cable have been on a cold day when it is 0°C? (use \( \alpha = 5.0 \times 10^{-3} / \text{°C} \))

\[ \rho = \rho_0 \left[ 1 + \alpha (T - T_0) \right] \]

From part (b) we have \( P = I^2 \rho \frac{L}{A} \),

\[ P = I^2 \rho_0 \left[ 1 + \alpha (T - T_0) \right] \frac{L}{A} \]
\[ P = \left[ 1 + \alpha (T - T_0) \right] P_0 \]
\[ P = \left[ 1 + \left(5.0 \times 10^{-3} \degree \text{C} \right) \left(0 \degree \text{C} - 20 \degree \text{C}\right) \right] \left(6.83378 \times 10^7 \text{ W}\right) \]
\[ P = 6.2 \times 10^7 \text{ W} \]

Or about 62 MW, a difference of about 10%.

3. In the circuit drawing to the right, \( V = 24.0 \text{V} \), \( R_1 = 10.0 \Omega \), \( R_2 = 8.00 \Omega \), \( R_3 = 5.00 \Omega \), \( R_4 = 12.0 \Omega \).

a) Find the current through the battery.

Make the equivalent circuits:
Adding the three remaining resistors in series we find that the equivalent resistance across the battery is 40.28Ω. Using Ohm’s Law:

\[ V_b = I_b R_{\text{equiv}} \quad \Rightarrow \quad I_b = \frac{V_b}{R_{\text{equiv}}} = \frac{24.0 \, \text{V}}{40.28 \, \text{Ω}} \]

\[ I_b = 0.60 \, \text{A} \]

b) Find the power expended by the battery and resistor \( R_4 \).

Since the current through the equivalent diagram is the same through all resistors \( I_b \) in the final diagram, we can find the power.

\[ P_b = I_b V_b = I_b^2 R_{\text{equiv}} = (0.60 \, \text{A})^2 (40.28 \, \text{Ω}) \]

\[ P_b = 14.5 \, \text{W} \]

\[ P_{R_4} = I_b^2 R_4 = 4.32 \, \text{W} \]

4. In the circuit drawing to the right, \( V_1 = 2.0\text{V}, V_2 = V_3 = 4.0\text{V}, R_1 = 1.00\text{Ω}, R_2 = 2.00\text{Ω}. \)

a) Find the three currents (i.e. the currents through the three batteries) in the circuit.

There are two junction points and two loops. It should be clear that the junction points \( A \) and \( B \) provide the same information. Let us denote the currents as \( I_{\text{left}}, I_{\text{right}} \) and \( I_{\text{middle}} \).

Taking Kirchhoff’s junction rule for point \( A \), let us have the current in the left and right branches flow inward and in the middle branch flow outward:

\[ I_f + I_r = I_m \]

Taking the two loops, we get:

\[ V_1 - I_f R_1 - I_m R_2 - V_2 - I_f R_1 = 0 \]

\[ V_1 - V_2 - 2I_f R_1 - I_m R_2 = 0 \]

and

\[ V_3 - I_f R_1 - I_m R_2 - V_2 - I_f R_1 = 0 \]

\[ V_3 - V_2 - 2I_f R_1 - I_m R_2 = 0 \]

Let us add these equations together.

\[ V_1 + V_3 - 2V_2 - 2(I_f + I_r) R_1 - 2I_m R_2 = 0 \]

From the junction rule we know \( I_f + I_r = I_m \)
Homework 5 - Solutions

\[ V_i + V_3 - 2V_2 - 2I_mR_i - 2I_mR_2 = 0 \]
\[ 2I_m(R_1 + R_2) = V_i + V_3 - 2V_2 \]

\[ I_m = \frac{V_i + V_3 - 2V_2}{2(R_1 + R_2)} = \frac{2.0V + 4.0V - 2(4.0V)}{2(1.0\Omega + 2\Omega)} \]

\[ I_m = -0.33\,\text{A} \]

The minus sign indicates that the flow is in the opposite direction of the original assumption.

We substitute the current for the middle back into the loop equations for the left and right sides

\[ V_1 - V_2 - 2I_iR_i - I_mR_2 = 0 \quad V_3 - V_2 - 2I_rR_i - I_mR_2 = 0 \]
\[ 2I_iR_i = V_1 - V_2 - I_mR_2 \quad 2I_rR_i = V_3 - V_2 - I_mR_2 \]

\[ I_i = \frac{V_1 - V_2 - I_mR_2}{2R_i} \quad I_r = \frac{V_3 - V_2 - I_mR_2}{2R_i} \]

\[ I_i = \frac{2.0V - 4.0V - (-0.3333\,\text{A})2.0\Omega}{2.0\Omega} \quad I_r = \frac{4.0V - 4.0V - (-0.3333\,\text{A})2.0\Omega}{2.0\Omega} \]

\[ I_i = -0.67\,\text{A} \quad I_r = 0.33\,\text{A} \]

Again, the minus sign for the current \( I_i \) means it is going in the opposite direction of the original assumption. So we have the currents as

left branch: \( 0.67\,\text{A} \) down

Middle branch: \( 0.33\,\text{A} \) up

Right branch: \( 0.33\,\text{A} \) up

b) Find the voltage difference between points A and B (\( V_{ab} \))

The difference in voltage between points A and B is found by finding the voltage drop through any path from A to B. Note that we use the correct direction for the current now!

Left path:

\[ V_{AB} = -I_iR_i - V_i - I_mR_2 = -(0.667\,\text{A})(1.0\Omega) - 2V - (0.667\,\text{A})(1.0\Omega) \]

\[ V_{AB} = -3.3\,\text{V} \]

Middle path:

\[ V_{AB} = +I_mR_2 - V_2 = + (0.333\,\text{A})(2.0\Omega) - 4V \]

\[ V_{AB} = -3.3\,\text{V} \]

Right path:

\[ V_{AB} = +I_rR_i - V_i + I_mR_1 = (0.667\,\text{A})(1.0\Omega) - 4V + (0.667\,\text{A})(1.0\Omega) \]

\[ V_{AB} = -3.3\,\text{V} \]
In all three cases we have $-3.3\text{V}$, which indicates that it is a drop of 3.3V from A to B.

5. In the circuit drawing to the right, $V = 1200\text{V}$, $C = 6.5\ \mu\text{F}$, $R_1 = R_2 = R_3 = 7.3 \times 10^5\ \Omega$. With C completely uncharged, the switch S is closed ($t = 0$)

**a)** **Determine the currents through each resistor for $t = 0$ and $t = \infty$.**

Use $V=q/C$ to solve with Kirchhoff’s rules with a capacitor. Let $I_1$, $I_2$, and $I_3$ be the currents through resistors $R_1$, $R_2$, and $R_3$ flowing as shown in the diagram.

$I_1$ will flow through the battery $V$ and $I_3$ will flow through the capacitor $C$.

Kirchhoff’s junction rule gives, $I_1 + I_2 = I_3$. The loop rules give the following.

$$V - I_1R_1 - I_2R_2 = 0 \quad \frac{q}{C} - I_3R_3 + I_2R_2 = 0$$

Since all resistors are the same:

$$V - I_1R - I_2R = 0 \quad \frac{q}{C} - I_3R + I_2R = 0$$

$$V - (I_1 - I_2)R = 0 \quad \frac{q}{C} + (I_2 - I_3)R = 0$$

At $t = 0$, we have no charge on the capacitor, so it provides no change in voltage:

$$V - (I_1 - I_2)R = 0 \quad (I_2 - I_3)R = 0$$

$$I_2 = I_3 \quad I_1 = 2I_2$$

$$V - (2I_2 - I_2)R = 0 \quad \Rightarrow \quad I_2 = \frac{V}{3R}$$

Which gives $I_2 = 0.55\ \text{mA}$, so, $I_3 = 0.55\ \text{mA}$ and $I_1 = 1.1\ \text{mA}$

At $t = \infty$, the capacitor is fully charged and there is no current flowing through it, so by definition $I_3 = 0$, so the junction rule gives $I_2 = I_1$ and only the lower loop is of concern.

$$V - I_1R - I_2R = 0 \quad \Rightarrow \quad V - 2I_1R = 0$$

$$I_1 = I_2 = \frac{V}{2R} = 0.82\ \text{mA}$$

**b)** **Determine the voltage across resistor $R_2$ for $t = 0$ and $t = \infty$.**
c) After a long enough period, the capacitor is fully charged. If the switch is now opened, how long will it take for the capacitor to reach 5% of its full charge?

\[
q = q_0 e^{-t/RC} \quad \Rightarrow \quad \frac{q}{q_0} = e^{-t/RC}
\]

\[
-\frac{t}{(R_2 + R_3)C} = \ln \left( \frac{q}{q_0} \right)
\]

\[
t = -(R_2 + R_3)C \ln \left( \frac{q}{q_0} \right) = -\left(2 \times 7.3 \times 10^2 \right) \left(6.5 \times 10^{-6} \text{ F} \right) \ln \left( \frac{0.05q_0}{q_0} \right)
\]

\[
t = 28 \text{ s}
\]