Chapter 19 – Examples

1. Two small metal spheres (with masses $m_A$ and $m_B$ and charges $q_A$ and $q_B$) are placed in a non-conducting, non-charged vertical tube. The lower sphere (A) is held in place, while sphere $B$ is free to move vertically but not horizontally. Initially, sphere $B$ is at rest and is above sphere $A$ by distance $d$.

   a) What is the maximum height sphere $B$ reaches above sphere $A$?

   We look at the conservation of energy, we are dealing with kinetic energy, gravitational potential energy and electric potential energy of sphere $B$:

   \[
   E_B = \frac{1}{2} m_B v_B^2 + m_B g y_B + q_B V_B
   \]

   \[
   E_B = \frac{1}{2} m_B v_B^2 + m_B g y_B + q_B \left( \frac{k q_A}{y} \right)
   \]

   where $y$ is the distance above sphere $A$. Since initially and at the highest point, the velocity of sphere $B$ will be zero, we need to find the potential energy of sphere $B$ at the lower and higher positions.

   The energy at the lowest and highest points for sphere $B$ must be equal:

   \[
   E_{low} = E_{high}
   \]

   \[
   q_B \left( \frac{k q_A}{y_{min}} \right) + m_Bgy_{min} = q_B \left( \frac{k q_A}{y_{max}} \right) + m_Bgy_{max}
   \]

   \[
   \frac{k q_A q_B}{d_l} + m_Bg d = \frac{k q_A q_B}{y_{max}} + m_Bgy_{max}
   \]

   \[
   y_{max} \left( \frac{k q_A q_B}{d_l} + m_Bg d \right) = k q_A q_B + m_Bg y_{max}^2
   \]

   This is a quadratic equation in terms of $y_{max}$ (the maximum height).

   \[
   m_Bgy_{max}^2 = \left( \frac{k q_A q_B}{d_l} + m_Bg d \right) y_{max} + k q_A q_B = 0
   \]

   \[
   y_{max} = \frac{\left( \frac{k q_A q_B}{d_l} + m_Bg d \right) \pm \sqrt{\left( \frac{k q_A q_B}{d_l} + m_Bg d \right)^2 - 4m_Bgk q_A q_B}}{2m_Bg}
   \]
\[ y_{max} = \frac{(kq_A q_B - m_B g d)}{2m_B g} \pm \sqrt{\frac{(kq_A q_B - m_B g d)^2}{2m_B g} + 2m_B g kq_A q_B + (m_B g d)^2} \]

Note that the terms under the radical form a perfect square!

\[ y_{max} = \frac{(kq_A q_B - m_B g d)}{2m_B g} \pm \sqrt{\left(\frac{kq_A q_B - m_B g d}{d_i}\right)^2 - 2m_B g kq_A q_B + (m_B g d)^2} \]

We now see two possible answers:

\[ y_{max} = \frac{kq_A q_B}{m_B g d_i} \text{ or } y_{max} = d \]

Obviously, the second answer occurs at the lower point, so the maximum height is

\[ y_{max} = \frac{kq_A q_B}{m_B g d_i} \]

Alternatively, you could choose \( y = 0 \) to be where sphere B starts, in which case, you must add the distance \( d \) to your final answer:

The energy at the lowest and highest points for sphere B must be equal:

\[ E_{low} = E_{high} \]

\[ q_B \left( \frac{kq_A}{y_{min} + d} \right) + m_B g y_{min} = q_B \left( \frac{kq_A}{y_{max} + d} \right) + m_B g y_{max} \]

\[ \frac{kq_A q_B}{d_i} = \frac{kq_A q_B}{y_{max} + d} + m_B g y_{max} \]

\[ (y_{max} + d) \frac{kq_A q_B}{d_i} = kq_A q_B + m_B g y_{max} (y_{max} + d) \]

\[ \frac{kq_A q_B}{d_i} y_{max} + kq_A q_B = kq_A q_B + m_B g y_{max}^2 + m_B g d y_{max} \]

\[ \left( \frac{kq_A q_B}{d_i} - m_B g d \right) y_{max} = m_B g y_{max}^2 \]

One value for \( y_{max} \) is zero, that is where sphere B starts.
\[
\frac{kq_A q_B}{d_i} y_{\text{max}} + kq_A q_B = kq_A q_B + m_B g y_{\text{max}}^2 + m_B g y_{\text{max}} d_i
\]

\[
\left( \frac{kq_A q_B}{d_i} - m_B g d_i \right) y_{\text{max}} = m_B g y_{\text{max}}^2
\]

One value for \( y_{\text{max}} \) is zero, that is where sphere B starts.

\[
y_{\text{max}} = \frac{kq_A q_B - m_B g d_i}{m_B g} = \frac{kq_A q_B}{m_B g} - d
\]

Adding \( d \) for the distance sphere B is above sphere A gives the same answer as above.

\[
y_{\text{max}} + d = \frac{kq_A q_B}{m_B g d_i}
\]

b) **Find the maximum velocity of sphere B.** (Hint: this occurs where the acceleration of sphere B is zero.)

Choosing \( y \) to be the distance above sphere A. The force of gravity and the electrostatic force on sphere B must be equal:

\[
F_g = m_B g \quad F_e = \frac{kq_A q_B}{y^2}
\]

\[
m g = \frac{kq_A q_B}{y^2}
\]

\[
y = \pm \sqrt{\frac{kq_A q_B}{m_B g}}
\]

The negative answer is obviously not allowed, so we have

\[
y = \sqrt{\frac{kq_A q_B}{m_B g}}
\]

Substitute back into the equation for total energy (1):

\[
E_B = \frac{1}{2} m_B v_{\text{max}}^2 + m_B g \sqrt{\frac{kq_A q_B}{m_B g}} + q_B \left( \frac{kq_A q_B}{\sqrt{m_B g}} \right)
\]

\[
E_B = \frac{1}{2} m_B v_{\text{max}}^2 + 2 \sqrt{kq_A q_B m_B g}
\]

Since this is the total energy, we can set it equal to the total energy at the lower point:

\[
m_B g d_i + q_B \left( \frac{kq_A q_B}{d_i} \right) = \frac{1}{2} m_B v_{\text{max}}^2 + 2 \sqrt{kq_A q_B m_B g}
\]

\[
\frac{1}{2} m_B v_{\text{max}}^2 = m_B g d_i + \frac{kq_A q_B}{d_i} - 2 \sqrt{kq_A q_B m_B g}
\]
\[ v_{\text{max}} = \sqrt{2gd + \frac{2kq_Aq_B}{m蒲} - 2\sqrt{\frac{kq_Aq_Bg}{m蒲}}} \]

Alternatively, you could choose \( y = 0 \) to be where sphere B starts, but the algebra is a little more complex. The answer will be the same, however.

c) Calculate the maximum height and maximum speed of sphere B if both spheres are electrons initially separated by 1.00 m.

Simply substitute the values into the answers for parts (a) and (b)

\[ y_{\text{max}} = \left( \frac{8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2}{(9.11 \times 10^{-31} \text{ kg})(9.81 \text{ m/s}^2)(1.00 \text{ m})} \right) \left( -1.602 \times 10^{-19} \text{ C} \right)^2 \]

\[ y_{\text{max}} = 25.8 \text{ m} \]

Likewise for maximum velocity

\[ v_{\text{max}} = 20 \text{ m/s} \]

2. A parallel plate capacitor consists of two square plates each 30.0 cm on a side and a separation of 1.0 mm. A neoprene rubber sheet (exactly 30.0 cm x 30.0 cm x 1.0 mm) is inserted between the plates. The plates are charged to a potential difference of 15kV. If there is no friction between the plates and the dielectric, how much work must be done to completely pull out the dielectric slab?

The capacitance of a parallel plate capacitor is given by eqn. 19.10:

\[ C = \frac{\kappa \varepsilon_0 A}{d} \]

The energy stored in a capacitor is \( \frac{1}{2} CV^2 \). The work done by pulling out the dielectric, must equal the change in energy stored by the dielectric:

\[ W = \text{Energy}_{\text{dielectric}} - \text{Energy}_{\text{no dielectric}} \]

\[ W = \frac{1}{2} C_d V^2 - \frac{1}{2} C_0 V^2 \]

\[ W = \frac{V^2}{2} \left( C_d - C_0 \right) \]
Often, an electron beam is used to investigate electric fields and potentials. In this case, a beam of electrons, each with velocity $v_0$, enters a parallel plate capacitor (length $l$, width $w$ and plate separation of $d$) parallel to the plates, but leaves with angle $\theta$ relative to the plane of the plates. (See figure to the right.)

a) What is the voltage difference of the two plates? (Ignore the edge effects of the plates and of gravity.)

The electric field in a parallel plate capacitor is perpendicular to the plates and constant. For an electron, with velocity $v_0$, to leave at an angle inclined to its original path, it must have a force applied in that direction. So, an electric field must exist between the plates that is perpendicular to the plates, such that its velocity perpendicular to the plates when leaving the plates is

$$\tan \theta = \frac{v_y}{v_0} \Rightarrow v_y = v_0 \tan \theta$$

where we have called the direction perpendicular to the plates ‘$y$’. The electric field only provides a force in the direction perpendicular to the plates so its initial velocity parallel to the plates is unaffected. Since the force is constant while the electron is between the plates, the acceleration is also constant:

$$a_y = \frac{F}{m} \quad \Rightarrow \quad v_y = a_y t$$

If we designate the moment the electron enters the field as $t = 0$, then time spent between the plates is

$$\ell = v_0 t \quad \Rightarrow \quad t = \frac{\ell}{v_0}$$

$$v_y = \frac{F}{m} \left( \frac{\ell}{v_0} \right)$$

We know that the force on a particle is $F = qE$ where the electric field is defined by eqn. 19.7a:
\[ E = \frac{\Delta V}{\Delta s} \]
\[ v_y = \frac{q\Delta V}{md} \left( \frac{\ell}{v_0} \right) \]

Equating this to (2)

\[ v_0 \tan \theta = \frac{q\ell \Delta V}{mv_0 d} \]
\[ \Delta V = \frac{v_0^2 md \tan \theta}{q \ell} \]

With the values for an electron:

\[ \Delta V = \frac{v_0^2 m_e d \tan \theta}{e \ell} \]

**b) Which plate is at the higher voltage?**

The upper plate (i.e. the one to which the electron is moving towards when inside the capacitor).