Qubits

- Qubits versus bits
- Requirements for a usable quantum computer
- Single qubit operations
- Two-qubits operations and quantum gates
- Charge qubits
- Flux qubits
- Spin qubits
Qubits versus bits

Why using quantum two-level systems (quantum bits = qubits) as fundamental logic units?

Classical logic: Boolean logic

Quantum logic: quantum superposition

\[ |0\rangle \text{ or } |1\rangle \quad \text{2 states} \quad \alpha |0\rangle + \beta |1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1 \quad \text{2 dimensions} \]

N bits: \(2^N\) states

N qubits: \(2^N\) dimensions

To present N qubits: M bits, \(M = 64 \cdot 2^N\)

Potential for novel computation algorithms:
Encryption, Fourier analysis, etc.

http://www-users.cs.york.ac.uk/~schmuel/comp/comp.html
Requirements for a usable q-computer

Physical realization of quantum computer shall provide

i) Qubits: *q. information is stored*

ii) Operation of the qubits: *q. information is processed*

iii) Read-out of the qubits: *q. information can be read*

iv) Weak decoherence: *no errors during computation*

v) Scalability: *put many qbits together*
We discuss some physical realizations of solid states qubits, together with specific operation, read-out and coupling schemes.

I) Josephson qubits

- phase qubits
- flux qubits
- charge qubits

II) Spin qubits
General description of qubits

One qubit: Analogy with s=1/2 particle in magnetic field

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \hat{H} |\psi\rangle$$

$$(\hat{\sigma}_z, \hat{\sigma}_x, \hat{\sigma}_y) \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Many qubits: pairwise couplings (6 per pair)

$$\hat{H} = \hbar (\vec{B}(t), \hat{\sigma})$$

3 fields to play

2x2 matrix

To switch on and off

Many qubits: pairwise couplings (6 per pair)
**Single qubit operations**

**Single-qubit operation**

a) Preparation = just wait a while...

Example: Spin 1/2 in a field $\mathbf{B}$ directed along $z$:

\[
|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

ground state \quad excited state

At $t = 0$ system will be in ground state if

\[
k_B T << \hbar |B_z|
\]
Single qubit operations II

b) Rotations

Apply a static field \( \vec{B}_{\text{ext}} = B_x \hat{x} \) for a time \( \tau \)

The spin will rotate following the laws of QM:

\[
|\psi(\tau)\rangle = U_x(\alpha)|\psi(0)\rangle
\]

where

\[
U_x(\alpha) = \begin{pmatrix}
\cos \alpha & i \sin \alpha \\
i \sin \alpha & \cos \alpha
\end{pmatrix}, \quad \alpha = B_x \tau
\]
Single qubits operations III

With \( |\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) it yields \( |\psi(\tau)\rangle = \begin{pmatrix} \cos \alpha \\ i \sin \alpha \end{pmatrix} \)

\[ \begin{aligned} \alpha = \pi/2 & \quad \text{Spin-flip (= NOT operation)} \\ \alpha = \pi/4 & \quad \text{linear superposition of} \\ & \quad \begin{pmatrix} \uparrow \rangle \\ \downarrow \rangle \end{pmatrix} \quad \text{with equal weight (= \( \sqrt{\text{NOT}} \))} \end{aligned} \]
c) Phase shifts

Apply a static field

\[ \vec{B}_{\text{ext}} = B_z \hat{z} \]

for a time \( \tau \)

\[ |\psi(\tau)\rangle = U_z(\beta) |\psi(0)\rangle \]

where

\[ U_z(\beta) = \begin{pmatrix} \exp(i\beta) & 0 \\ 0 & \exp(-i\beta) \end{pmatrix}, \quad \beta = B_z \tau \]
d): Resonant driving ($B_z$ present, splits the levels)
Can be done with small time-dependent rotating field

$$\vec{B}_{\text{ext}}(t) = B_R \cos(\Omega t) \hat{x} + B_R \sin(\Omega t) \hat{y}$$

$$H(t) = B_z \sigma_z + B_R [(\cos \Omega t) \sigma_x + (\sin \Omega t) \sigma_y]$$

- choose frequency in resonance
  $$\Omega = 2B_z$$

- transform to rotating frame
  $$|\psi\rangle \rightarrow R_{\Omega} |\psi\rangle; R_{\Omega} = \exp(-i\Omega \sigma_z t / 2)$$

$$\tilde{H} \approx B_R(t) \sigma_x$$

Looks like static Hamiltonian with $B_z$ cancelled:
Apply previous techniques
Quantum gates

A quantum gate realizes a unitary operation with N qubits

• Number of bits in equals number of bits out
• The operation is reversible (in contrast to usual computation)

Example of a two-bit (quantum) gate: CNOT

\[
\begin{pmatrix}
|0\rangle|0\rangle & \rightarrow & |0\rangle|0\rangle \\
|0\rangle|1\rangle & \rightarrow & |0\rangle|1\rangle \\
|1\rangle|0\rangle & \rightarrow & |1\rangle|1\rangle \\
|1\rangle|1\rangle & \rightarrow & |1\rangle|0\rangle
\end{pmatrix}
\]

Negation of the second (q)bit provided the first is in the state |1\rangle

Math says: CNOT is enough for arbitrary q. algorithm

\[
U_{CNOT} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

in the basis \{\langle \uparrow\uparrow\rangle, \langle \uparrow\downarrow\rangle, \langle \downarrow\uparrow\rangle, \langle \downarrow\downarrow\rangle\}
Quantum gates II

How to realize two-qubit gates using unitary transformations?

Make pulses of the qb-qb coupling

Example:

$$H_{\text{int}} = J (\sigma_x^{(1)} \sigma_x^{(2)} + \sigma_y^{(1)} \sigma_y^{(2)})$$

in the basis \{\text{\(\uparrow\uparrow\), \(\uparrow\downarrow\), \(\downarrow\uparrow\), \(\downarrow\downarrow\)}\}

The resulting two-bit operation is controlled by pulse duration

$$U_{12}(\gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos \gamma & i \sin \gamma & 0 & 0 \\ 0 & i \sin \gamma & \cos \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = 2J \tau / \hbar$$
Quantum gates III

Usually one has to combine (several) two-bit operations ($U$) and (several) single-bit rotations ($R$)

\[
U_{\text{CNOT}} = R_2^x (\pi / 2) R_2^z (-\pi / 2) R_2^x (-\pi) \\
\times U_{12} (-\pi / 2) \\
\times R_1^x (-\pi / 2) \\
\times U_{12} (\pi / 2) \\
\times R_1^z (-\pi / 2) R_2^z (-\pi / 2)
\]
Some physical realizations

|                | Spin 1/2 $|\uparrow\rangle,|\downarrow\rangle$ | Charge qubits $|0\rangle,|1\rangle$ | Flux qubits $|L\rangle,|R\rangle$ |
|----------------|---------------------------------|---------------------------------|---------------------------------|
| system $H_S$   | $-\frac{\hbar}{2} gB_z \sigma_z$ | $-\frac{\varepsilon(n_g)}{2} \sigma_z - \frac{\Delta(\Phi_{ext})}{2} \sigma_x$ | $-\frac{\varepsilon(\Phi_{ext})}{2} \sigma_z - \frac{\Delta}{2} \sigma_x$ |
| control fields | $\propto \sigma_z, \sigma_x$     | $\propto \sigma_z, \sigma_x$     | $\propto \sigma_z$             |
| coupling       | $\propto \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}$ | $\propto \sigma_y^{(1)} \sigma_y^{(2)}$ or $\sigma_z^{(1)} \sigma_z^{(2)}$ | $\propto \sigma_z^{(1)} \sigma_z^{(2)}$ |
| Decoherence    | $\propto \sigma_z L$             | $\propto \sigma_z L+T$          | $\propto \sigma_z L+T$          |

$L = $ longitudinal, $T = $ transverse

Table updated to 31.07.2003
Refreshing Josephson q.m.

- Phase (flux) representation

\[ E\Psi(\varphi) = \hat{H}\Psi = (E_C (-2i \frac{\partial}{\partial \varphi} - \frac{Q_g}{e})^2 - E_J \cos \varphi)\Psi \]

- Charge representation

\[ E\Psi(N) = \hat{H}\Psi = E_C (N - \frac{Q_g}{e})^2 \Psi(N) - \frac{E_J}{2}(\Psi(N - 2) + \Psi(N + 2)) \]

- Boundary conditions- single junction or single island

\[ E_C = \frac{e^2}{2C} \]
Phase qubits

i) The system: current biased large JJ \( E_J / E_C >> 10^4 \)

- Phase \( \phi \) is a coordinate (conjugate to continuous charge through capacitor)
- Josephson potential
  \[
  E(\phi) = E_J \cos \phi - I_b \phi \]
  \( I_b \sim I_c \)

- several q.states formed

\[ |0\rangle, |1\rangle \ used \ as \ qubit \ states \]

*Martinis et al., PRL 89, 117901 (2002)*
ii) Manipulation

Apply resonant microwaves with $E_{10} = E_1 - E_0$

Rabi oscillations

iii) Read-out

Apply microwave pulse resonant with $E_{21} = E_2 - E_1$

Single shot measure of $P_1(t)$

iv) Quantum gate: Capacitive JJ -JJ coupling (not yet demonstrated)
Flux qubits \( \phi \equiv 2\pi \Phi / \Phi_0 \)

i) The system: rf-SQUID

\[
H = -E_j^0 \cos(2\pi \Phi / \Phi_0) + \frac{(\Phi - \Phi_{\text{ext}})^2}{2L} + \frac{Q^2}{2C_j^0}
\]

charge \( Q \), flux \( \Phi \): canonically conjugated

Parameter range:
\[
\begin{align*}
E_j^0 & >> E_C^0 \\
\frac{E_j^0}{\Phi_0^2} & \approx L \\
\Phi & \approx \Phi_0 / 2
\end{align*}
\]

Positive and negative inductances match

! a double-well potential centered at \( \Phi \sim \Phi_0 / 2 \) with energy shift

\( \hbar \epsilon \sim (\Phi_{\text{ext}} / \Phi_0 - 1/2) \)
Flux qubits II

Level splitting in symmetric double-well

\[ \hbar \Delta_n = E_{2n+2} - E_{2n+1} \]

Affected by asymmetry

\[ E = \hbar \sqrt{\Delta^2 + \varepsilon^2} \]

Truncation to two-states

\[ H_{\text{TLS}} = -\frac{\hbar}{2} (\varepsilon \sigma_z + \Delta \sigma_x) \]

\[ \sigma_z |L\rangle = -|L\rangle \quad \sigma_z |R\rangle = |R\rangle \]
Flux qubits III

ia) The improved system: 3-junction Delft-SQUID

\[ \Phi_3, \tilde{E}_J^0 \]
\[ \Phi_1, E_J^0 \]
\[ \Phi_2, E_J^0 \]

Requirements:

- \( E_J \geq E_C \) \( \rightarrow \) Flux=phase well-defined
- truncation to the lowest two states

\[ H_{\text{TLS}} = -\frac{\hbar}{2} (\epsilon \sigma_z + \Delta \sigma_x) \]

\( \sigma_z |L\rangle = -|L\rangle \)
\( \sigma_z |R\rangle = |R\rangle \)

\( \sigma_z \) “position”

less sensitive to external noise than rf-SQUID


Flux qubits IV

ii) Manipulation: resonant ac-control fields, dc pulses \( \propto \sigma_z \)

\[ \text{Rabi oscillations, Ramsey interference, echo} \]

iii) Read-out: coupling to a dc-squid

\[ \text{variation in SQUID switching current sensitive to qubit state} \]

iv) Inductive qubit-qubit coupling

\[ H_{\text{int}} \propto \sigma_z^{(1)} \sigma_z^{(2)} \]

v) Dissipation proportional to \( \sigma_z \) (flux noise)

\[ \text{longitudinal + transverse components in energy basis} \]
Charge qubits

i) The system: the Cooper pair box (Charging energy dominates)

Josephson energy is tunable

\[ E_J(\Phi_{\text{ext}}) = 2E_J^0 \cos(\pi \Phi_{\text{ext}} / \Phi_0) \]

Charge balance affected by gate

Yu. Makhlin, G. Schön and A. Shnirman, Nature 368, 305 (1999); Rev. Mod. Phys. 73, 357 (2001)


Charge qubits II

Requirements

I) $\Delta_s \gg E_C \gg E_J$
- only Cooper pairs tunnel through the junctions
- number $n$ of charges is well defined

$\Delta_s$ superconducting gap

II) Choose gate voltage ($n_g$) so that the parabolas cross
- only charge states $n = 0$, $n = -2$ are relevant

$E(n, n_g)$

$n = 0$
$n = -1$
$n = -2$

$E_J \approx 0$

$n_g$
Charge qubits III

\[ \sigma_z |n = 0\rangle = - |n = 0\rangle, \quad \sigma_z |n = -2\rangle = |n = -2\rangle \]

qubit Hamiltonian

\[ H_{TLS} = -\frac{\hbar}{2} [\epsilon(n_g)\sigma_z + \Delta(\Phi_{ext})\sigma_x] \]

\[ \hbar \epsilon = 4E_C(2n_g - 1), \quad \Delta = E_J(\Phi_{ext}) \]

In the representation \( |0\rangle, |1\rangle \) \( H_{TLS} \) is non diagonal unless \( \Delta = 0 \)
Experimental realization

ii) Manipulation: nonadiabatic dc pulses

Rabi oscillations

iii) Read out: quasi-particle tunneling through probe junction (Too bad)
Experimental observation

Rabi oscillations

experiment

theory

Coupling of charge qubits

iva) Inductive qubit-qubit coupling

\[ H_{\text{int}} \propto \sigma_y^{(1)} \sigma_y^{(2)} \]


ivb) Capacitive coupling

\[ H_{\text{int}} \propto \sigma_z^{(1)} \sigma_z^{(2)} \]


iv) Dissipation proportional to \( \sigma_z \) (charge noise + background charges)

longitudinal + transverse components in energy basis
Spin qubits

i) Spin degree of freedom $\sigma$ of an electron in a quantum dot

$$H_M = -\frac{\hbar}{2} \gamma \vec{\sigma} \cdot \vec{B}$$

$\gamma$ giromagnetic ratio

$\sigma_x, \sigma_y, \sigma_z$, Pauli matrices

Choose: $\vec{B} = B_z \hat{z}$

$$H_M = -\frac{\hbar}{2} \gamma B_z \sigma_z$$

is diagonal

Two coupled qubits

where $\varepsilon = \gamma B_z$
Spin qubits II

ii) Manipulation: (resonant) external field \( \vec{B}_x(t) \)

iii) Read out:
Spin to charge,
Charge modulates
Current through point contacts (PC)

iii) Tunable barrier: tunable exchange coupling \( \propto \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} \)

iv) Dissipation is longitudinal \( \propto \sigma_z + \text{spins} + \ldots \)