Gauss's Law in Pictures

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PHYS 208 Honors: Fundamentals of Physics II
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A charge distribution is symmetric if there is a set of geometrical transformation that do not cause any physical change.
The symmetry of the electric field must match the symmetry of the charge distribution.

(a) Is this a possible electric field of an infinitely long charged cylinder? Suppose the charge and the field are reflected in a plane perpendicular to the axis.

(b) The charge distribution is not changed by the reflection, but the field is. This field doesn’t match the symmetry of the cylinder, so the cylinder’s field can’t look like this.
Symmetry of Charge Distribution vs. Symmetry of Electric Field

The symmetry of the electric field must match the symmetry of the charge distribution.

(a) End view of cylinder

Reflection plane

The charge distribution is not changed by reflecting it in a plane containing the axis.

Reflect

(b) This field is changed. It doesn’t match the symmetry of the cylinder, so the field can’t look like this.
Electric Field Pattern of Cylindrically Symmetric Charge Distribution

Side view

End view

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Electric Field Pattern of Charge Distributions with Basic Symmetries

Planar symmetry

The field is perpendicular to the plane.

Cylindrical symmetry

The field is radial toward or away from the axis.

Spherical symmetry

The field is radial toward or away from the center.
The electric field of a cylindrically symmetric charge distribution cannot have a component parallel to the cylinder axis. Also, the electric field of a cylindrically symmetric charge distribution cannot have a component tangent to the circular cross section. The only shape for the electric field that matches the symmetry of the charge distribution with respect to (i) translation parallel to the cylinder axis, (ii) rotation by an angle about the cylinder axis, and (iii) reflections in any plane containing or perpendicular to the cylinder axis is the one shown in the figure.
Example: Problem 27.2
Example: Problem 27.3

\[ \vec{E} = 0 \text{ N/C} \]
The Concept of Flux of Electric Field

(a) The field is coming out of each face of the box. There must be a positive charge in the box.

(b) The field is going into each face of the box. There must be a negative charge in the box.

(c) A field passing through the box implies there’s no net charge in the box.

(a) A Gaussian surface is a closed surface around a charge.

(b) A two-dimensional cross section through a spherical Gaussian surface is easier to draw.
Gaussian surface which does not match symmetry of charge is not useful!

The electric field pattern through the surface is particularly simple if the closed surface matches the symmetry of the charge distribution inside.
Calculus for the Flux of Vector Fields

Motivation for the definition: Flux of a fluid flow

(a) Loop
The air flowing through the loop is maximum when $\theta = 0^\circ$.

(b) Unit vector normal to loop
No air flows through the loop when $\theta = 90^\circ$.

(c) The loop is tilted by angle $\theta$.
$v_\perp = v \cos \theta$ is the component of the air velocity perpendicular to the loop.

Flux of uniform electric field passing through a surface:

$E_\perp = E \cos \theta$ is the component of the electric field that passes through the surface.

Normal to surface
$\theta$ is the angle between $\hat{n}$ and $\vec{E}$.

(a) Area vector $\vec{A}$ is perpendicular to the surface. The magnitude of $\vec{A}$ is the surface area $A$.

(b) The electric flux through the surface is $\Phi_e = \vec{E} \cdot \vec{A}$.
Flux of a Nonuniform Electric Field

Simplified situations where integration goes away:

- **(a)**
  - $\vec{E}$ is everywhere tangent to the surface. The flux is zero.

- **(b)**
  - $\vec{E}$ is everywhere perpendicular to the surface and has the same magnitude at each point. The flux is $EA$. 

$$\Phi_e = \sum_{i=1}^{N} \delta \Phi_i = \sum_{i=1}^{N} \vec{E} \cdot (\delta \vec{A})_i$$

$$\Phi_e = \int_{\text{surface}} \vec{E} \cdot d\vec{A}$$

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**Gauss’s Law For Point Charge**

Cross section of a Gaussian sphere of radius \( r \). This is a mathematical surface, not a physical surface.

The electric field is everywhere perpendicular to the surface and has the same magnitude at every point.

Point charge \( q \)

\[
\Phi_e = \oint \vec{E} \cdot d\vec{A} = \int_{\Sigma} \vec{E} \cdot d\vec{A} = EA_{Sphere} = E4\pi r^2
\]

\[
\Phi_e = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} 4\pi r^2 = \frac{q}{\varepsilon_0}
\]

Electric flux is independent of surface shape:

\[
d\Phi_e = EdA\cos \alpha = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} dA\cos \alpha = \frac{q}{4\pi\varepsilon_0} d\Omega
\]

Every field line passing through the smaller sphere also passes through the larger sphere. Hence the flux through the two spheres is the same.

The spherical pieces can slide in or out to form a complete sphere. Hence the flux through the pieces is the same as the flux through a sphere.
Gauss’s Law For Charge Distribution = First Maxwell Equation

Unlike Coulomb’s law for static point charges, Gauss’s law is valid for moving charges and fields that change with time.
Applications: Charged Sphere

1. Determine the symmetry of the electric field.

2. Select a Gauss surface (as imaginary surface in the space surrounding the charge) to match the symmetry of the field.

3. If 1. and 2. are possible, flux integral should become algebraic product electric field x Gauss surface area.

\[ \Phi_e = \oint \vec{E} \cdot d\vec{A} = E_{\text{outside}} A_{\text{sphere}} = \frac{Q}{\varepsilon_0} \]

\[ E_{\text{outside}} 4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow \vec{E}_{\text{outside}} = \frac{q}{4\pi\varepsilon_0 r^2} \hat{r} \]

\[ \Phi_e = \oint \vec{E} \cdot d\vec{A} = E A_{\text{sphere}} = \frac{Q_{\text{inside}}}{\varepsilon_0} \]

\[ E_{\text{inside}} 4\pi r^2 = \frac{4}{3} \frac{r^3}{\varepsilon_0} \pi \rho = \frac{4}{3} \frac{r^3}{\varepsilon_0} \pi \frac{Q}{4/3 R^3 \pi} \Rightarrow \vec{E}_{\text{inside}} = \frac{Q}{4\pi\varepsilon_0 R^3} r \hat{r} \]
Applications: Charged Wire

The field is tangent to the surface on the ends. The flux is zero.

The field is perpendicular to the surface on the cylinder wall.

$$\Phi_e = \oint \vec{E} \cdot d\vec{A} = \Phi_{top} + \Phi_{bottom} + \Phi_{wall} = 0 + 0 + EA_{cylinder}$$

$$E \frac{2\pi rL}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0} \Rightarrow E_{wire} = \frac{Q}{2\pi \varepsilon_0 r} = \frac{\lambda}{2\pi \varepsilon_0 r}$$
Applications: Charged Plane

\[ \Phi_e = \oint \vec{E} \cdot d\vec{A} = 2EA \]

\[ EA = \frac{Q_{in}}{\varepsilon_0} = \frac{\eta A}{\varepsilon_0} \Rightarrow E_{plane} = \frac{\eta}{2\varepsilon_0} \]
BASIC FACT: Electric field is zero at all points within the conductor - otherwise charges will flow, thereby violating electrostatic equilibrium.

\[ \Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{inside}}}{\varepsilon_0} \Rightarrow Q_{\text{inside}} \equiv 0 \]

Any excess charge resides entirely on the exterior surface.

Any component of the electric field tangent to the conductor surface would cause surface charges to move thereby violating the assumption that all charges are at rest.
Gauss's Law Determines Electric Field Outside of Metal

Surface charge density $\eta$

The electric field is perpendicular to the surface.

The electric field $E = 0$

\[ \Phi_e = E_{\text{surface}} A = \frac{Q_{\text{in}}}{\varepsilon_0} = \frac{\eta A}{\varepsilon_0} \Rightarrow E_{\text{surface}} = \frac{\eta}{\varepsilon_0} \hat{n} \]
Electric Field of Conducting Shells

A hollow completely enclosed by the conductor

\[ \vec{E} = 0 \]

The flux through the Gaussian surface is zero. There’s no net charge inside, hence no charge on this interior surface.

The flux through the Gaussian surface is not zero. \[ \oint \vec{E} \cdot d\vec{S} \neq 0 \]

The outer surface must have charge \( +q \) in order that the conductor remain neutral.

The flux through the Gaussian surface is zero, hence there’s no net charge inside this surface. There must be charge \(-q\) on the inside surface to balance point charge \( q \).
Applications: Faraday's Cage

(a) Parallel-plate capacitor

We want to exclude the electric field from this region.

(b) The conducting box has been polarized and has induced surface charges.

The electric field is perpendicular to all conducting surfaces.
Example: Electric Force Acting on the Surface of Metallic Conductor

\[ \vec{E} = 0 \Rightarrow E_{\delta A} = E_{\text{rest}} \]

\[ E_{\text{surface}} = E_{\delta A} + E_{\text{rest}} = 2E_{\text{rest}} \Rightarrow E_{\text{rest}} = \frac{E_{\text{surface}}}{2} \]

\[ \vec{F}_{\delta A} = \eta \delta A \frac{\vec{E}_{\text{surface}}}{2} \quad \iff \quad \vec{f} = \frac{\vec{F}_{\delta A}}{\delta A} = \frac{1}{2} \eta \vec{E}_{\text{surface}} \]

\[ \vec{f} = \frac{1}{2} \varepsilon_0 E_{\text{surface}}^2 \hat{n} \Rightarrow \vec{F}_{\text{conductor}} = \frac{\varepsilon_0}{2} \oint E_{\text{surface}}^2 \, d\vec{A} \]

NOTE: Regardless of the sign of surface charge density \( \eta \) and corresponding direction of the electric field \( E_{\text{surface}} \), the force \( \vec{F}_{\text{surface}} \) is always directed outside of the conductor tending to stretch it.