Chapter 6 Sommerfeld Theory of Metals

Failure of Drude model.

Specific heat of electrons

Drude predicted: \( C_V = \frac{3}{2} k_B n \)

Experiment: \( C_V \) vanishes as \( T \rightarrow 0 \)

Origin of the problem: electrons was treated as classical gas molecules.

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<td>( f_B(\vec{v}) = n \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{mv^2}{2k_B T}\right) )</td>
<td>( f(\vec{v}) = \left( \frac{m}{\pi k_B T} \right)^{3/2} \exp\left(-\frac{mv^2}{k_B T}\right) )</td>
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<td>Average velocity</td>
<td>( \frac{1}{2} m v_0^2 = \frac{3}{2} k_B T )</td>
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Fermi-Dirac distribution implied Pauli exclusion principle.

High density implies spatial confinement.

The combination of these two factors result in a substantial kinetic energy of electrons even at \( T = 0 \).
A electron wave packet has a spread in real space as well as in momentum space.

\[ \Delta p \cdot \Delta x \geq \frac{\hbar}{2} \]

\[ \Delta p \cdot \Delta x \sim h \]

\[ h = \frac{\hbar}{2\pi} \]

When \( \Delta x \) is small (in the case of high particle density \( n \)), \( \Delta p \) has to be substantial.

\[ \Delta p \sim p \]

Thus kinetic energy \( E_k = \frac{p^2}{2m} \) is substantial. Therefore, high density electron gas is energetic even at \( T=0 \).

Consider one dimensional electron gas confined between \( -\frac{a}{2} < x < \frac{a}{2} \) in phase space.

\[ \text{area} \ h \]

\[ x \]

\[ p \]
Partition the phase space into squares (or rectangles) of area $\hbar$. Each little rectangle represents an electron orbital wave packet ($\Delta p \cdot \Delta x = \hbar$).

Due to Pauli exclusion principle, only two electrons with opposite spins can occupy each rectangle.

For a large number of electrons $N$, electrons have to pile up from $p = 0$ to $|p| = p_F$.

Area of occupied phase space:
$$2p_F \cdot a.$$

Number of occupied orbital states:
$$\frac{2p_F \cdot a}{\hbar}.$$

Total number of electrons in occupied states (Each orbital state accommodates two electrons)
$$2 \times \frac{2p_F \cdot a}{\hbar} = N$$

$$p_F = \frac{N \cdot \hbar}{4a}$$ \hspace{1cm} A measure of electron momentum at $T=0$

$$\hbar k_F = \frac{N}{4a} \cdot 2\pi \hbar$$
\[ k_F = \frac{\pi N}{2a}. \quad \text{The highest wave vector (Fermi wave vector).} \]

Density of states: number of orbital electron states per unit length of \( k \)

\[ \frac{N/2}{2k_F} = \frac{N/2}{\frac{\pi N}{a}} = \frac{a}{2\pi} \]

Three dimensional density of states:

\[ \left( \frac{a}{2\pi} \right)^3 = \frac{V}{8\pi^3} \]
Ground State Properties of Electron Gas

* Assumptions:
  a. Consider the ground-state properties of $N$ electrons confined to a volume $V$
  b. Electrons do not interact with one another.

One can find the ground state of $N$ electrons by first finding the energy levels of a single electron in volume $V$, and then filling these levels up in a manner consistent with the Pauli exclusion principle.

A single electron wave function: $\psi(\vec{r})$

\[-\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi(\vec{r}) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) = \varepsilon \psi(\vec{r}).\]

Confinement of electron: cube of side $L = V^{1/3}$

Use periodic (Born-von Karman) boundary condition.

Solution to Schrödinger equation:

\[\psi_k(\vec{r}) = \frac{1}{\sqrt{V}} e^{i \vec{k} \cdot \vec{r}}\]

with energy \(\varepsilon(\vec{k}) = \frac{\hbar^2 k^2}{2m}\)

factor $\frac{1}{\sqrt{V}}$ comes from normalization $| = \int d\vec{r} |\psi(\vec{r})|^2$
\( \Psi_k \) is an eigenstate of the momentum operator:

\[ \vec{p} = \frac{\hbar}{i} \frac{\partial}{\partial \vec{r}} = \frac{\hbar}{i} \nabla \quad \left( \vec{p}_x = \frac{\hbar}{i} \frac{\partial}{\partial x}, \text{ ect.} \right) \]

with eigenvalue \( \vec{p} = \hbar \vec{\hat{k}} \)

\[ \frac{\hbar}{i} \frac{\partial}{\partial \vec{r}} e^{i \vec{k} \cdot \vec{r}} = \hbar \vec{\hat{k}} e^{i \vec{k} \cdot \vec{r}} \]

momentum of electron: \( \vec{p} = \hbar \vec{\hat{k}} \)

velocity of electron: \( \vec{v} = \frac{\vec{p}}{m} = \frac{\hbar}{m} \vec{\hat{k}} \)

wavelength of electron: \( \lambda = \frac{2\pi}{k} \)

Apply the boundary condition:

\[ e^{i k_x (x + L)} = e^{i k_x x} \]

\[ e^{i k_y (y + L)} = e^{i k_y y} \]

\[ e^{i k_z (z + L)} = e^{i k_z z} \]

\[ e^{i k_x L} = e^{i k_y L} = e^{i k_z L} = 1 \]

\[ k_x L = 2\pi n_x \quad k_y L = 2\pi n_y \quad k_z L = 2\pi n_z \]

\[ k_x = \frac{2\pi n_x}{L} \quad \quad k_y = \frac{2\pi n_y}{L} \quad \quad k_z = \frac{2\pi n_z}{L} \]

\( n_x, n_y, n_z \) all integers
In two dimensional $k$-space,

area per point \( \left( \frac{2\pi}{L} \right)^2 \)

In three dimensional $k$-space

area per point \( \left( \frac{2\pi}{L} \right)^3 \)

A region of $k$-space of volume $V$ contains

\[
\frac{\sqrt{V}}{\left( \frac{2\pi}{L} \right)^3} = \frac{RV}{8\pi^3}
\]

Allowed values of $\vec{k}$

Equivalently, the number of allowed $k$-values per unit volume of $k$-space:

\[
\frac{V}{8\pi^3}
\]

known as the $k$-space density of levels.

Fill up the $N$-electron ground state in $\vec{k}$-space

\[
\begin{align*}
\text{Start from lowest energy level } (\epsilon = 0) \\
\text{Two electrons (spin-up and spin-down) on each level.} \\
\text{Fill up to Fermi level } |\vec{k}| = k_F
\end{align*}
\]
This results in a sphere in k-space with radius $k_F$.

Volume: $\mathcal{V} = \frac{4}{3} \pi k_F^3$

number of allowed $k$-values with sphere:

$\left(\frac{4}{3} \pi k_F^3 \frac{V}{8\pi^3}\right) = \frac{k_F^3}{6\pi^2} V$

In order to accommodate $N$ electrons:

$N = 2 \cdot \frac{k_F^3}{4\pi^2} V = \frac{k_F^3}{3\pi^2} V$

If we have $N$ electrons in a volume $V$ (i.e. an electronic density $n = N/V$), then the ground state of $N$-electron system is formed by occupying all single-particle levels with $k$ less than $k_F$, and leaving all those with $k$ greater than $k_F$ unoccupied, where $k_F$ is given by

$N = \frac{k_F^3}{3\pi^2}$

sphere of radius $k_F$: Fermi sphere

Surface of Fermi sphere: Fermi surface

momentum $p_F = \hbar k_F$: Fermi momentum

$E_F = \frac{\hbar^2 k_F^2}{2m}$: Fermi energy

$= \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}$

$V_F = \frac{p_F}{m}$: Fermi velocity
\[
\frac{1}{n} = \frac{4\pi}{3} \pi r_s^3
\]

\[
k_F = (3\pi^2 n)^{\frac{1}{3}} = \left(\frac{9\pi}{4}\right)^{\frac{1}{3}} = \frac{1.92}{r_s} = \frac{3.63}{r_s/a_0} \text{ Å}^{-1}
\]

\[
\gamma_s/a_0 \sim 2-6 \quad \text{in metallic elements}
\]

\[
U_F = (\frac{\hbar}{m}) k_F = \frac{4.20}{r_s/a_0} \times 10^8 \text{ cm/sec}
\]

A substantial velocity

\[
U_F \sim 10^8 \text{ cm/s} \quad \text{even at } T=0
\]

In classical gas, \( u=0 \) at \( T=0 \)

Even at RT, the thermal velocity of a classical particle with electronic mass is only \( \sim 10^7 \text{ cm/s} \)

\[
E_F = \frac{\hbar^2 k_F^2}{2m}, \quad a_0 = \frac{\hbar^2}{me^2}
\]

\[
E_F = \left(\frac{e^2}{2a_0}\right) \left( k_F a_0 \right)^2
\]

\[
\frac{e^2}{2a_0} = 13.6 \text{ eV} : \text{Rydberg (Ry), the ground state binding energy of hydrogen atom}
\]

\[
a_0 = 0.529 \times 10^{-8} \text{ cm}
\]

\[
E_F = \frac{50.1 \text{ eV}}{(r_s/a_0)^2} \sim 1.5 - 15 \text{ eV for metallic elements}
\]
Total Ground State energy of $N$ electrons in a volume $V$

$$E = 2 \sum_{k<k_F} \frac{\hbar^2 k^2}{2m}$$

the volume of $k$-space per allowed $k$ value is

$$\Delta \bar{k} = \frac{8\pi^3}{V}$$

$$E = 2 \cdot \frac{V}{8\pi^3} \sum_{k<k_F} \frac{\hbar^2 k^2}{2m} \Delta \bar{k}$$

$$E = 2 \cdot \frac{V}{8\pi^3} \int_{k<k_F} \frac{\hbar^2 k^2}{2m} d\bar{k}$$

$$\frac{E}{V} = \frac{1}{4\pi^3} \int_{k<k_F} \frac{\hbar^2 k^2}{2m} d\bar{k} = \frac{1}{\pi^2} \frac{\hbar^2 k_F^5}{10m}$$

using $\frac{N}{V} = \frac{k_F^3}{3\pi^2}$

$$\frac{E}{N} = \frac{3}{10} \frac{\hbar^2 k_F^2}{m} = \frac{3}{5} E_F$$

or $\frac{E}{N} = \frac{3}{5} k_B T_F$

$$T_F = \frac{E_F}{k_B} = \frac{58.2}{(r_s/a_0)^2} \times 10^4 \text{ K} \quad \text{Fermi temperature}$$
Thermal Properties of the free electron gas.

The Fermi–Dirac distribution.

\( f_i^N \) : the probability of there being an electron in the particular one-electron level \( i \), when the \( N \) electron system is in thermal equilibrium.

Fermi–Dirac distribution

\[
f_i^N = \frac{1}{e^{(E_i - \mu)/k_BT} + 1}
\]

\[
N = \sum_i f_i^N = \sum_i \frac{1}{e^{(E_i - \mu)/k_BT} + 1}
\]

\( \mu \): chemical potential.

\( \epsilon(E) = \frac{1}{2m} \frac{\hbar^2 k^2}{E} \)

At \( T = 0 \):

\[
\lim_{T \to 0} \frac{f_{KS}}{E} = 1 \quad \epsilon(E) < \mu.
\]

\[
\lim_{T \to 0} = 0 \quad \epsilon(E) > \mu.
\]

To be consistent with the ground state picture,

\[
\lim_{T \to 0} \mu = \epsilon_F
\]
At Finite Temperature:

Internal energy \( U = 2 \sum \frac{E(k)}{k} \int f(E(k)) \Delta \vec{k} \)

\[
f(E) = \frac{1}{\exp \left( \frac{E - \mu}{k_B T} \right) + 1}
\]

\[
\Delta \vec{k} = \frac{8\pi^2}{\sqrt{V}}
\]

\[
U = 2 \cdot \frac{V}{8\pi^3} \sum_k \frac{E(k)}{k} \int f(E(k)) \Delta \vec{k}
\]

\[
\frac{U}{V} = \frac{V}{4\pi^3} \int \frac{E(k)}{k} \int f(E(k)) d\vec{k}
\]

\[
N = \sum f_i = \sum_i \frac{1}{\exp \left( \frac{E_i - \mu}{k_B T} \right) + 1}
\]

\[
= 2 \cdot \frac{V}{8\pi^3} \sum_k \frac{1}{\exp \left( \frac{E - \mu}{k_B T} \right) + 1} \Delta \vec{k}
\]

\[
= \frac{V}{4\pi^3} \int \frac{d\vec{k}}{k} \int f(E(k))
\]

\[
\beta = \frac{N}{V} = \int \frac{d\vec{k}}{4\pi^3} f(E(k))
\]

In evaluating integrals of the form

\[
\int \frac{d\vec{k}}{4\pi^3} f(E(k))
\]

One can exploit the fact that the integrand depends on \( k \) only through the electronic energy \( E = \frac{\hbar^2 k^2}{2m} \)
\[
\int \frac{d^{3}k}{4\pi^{3}} \tilde{F}(\varepsilon(\mathbf{k})) = \int_{0}^{v_{0}} \frac{4\pi k^{2}dk}{4\pi^{3}} \tilde{F}(\varepsilon(\mathbf{k}))
\]
\[
= \int_{0}^{v_{0}} \frac{k^{2}dk}{\pi^{2}} \tilde{F}(\varepsilon(\mathbf{k}))
\]
\[
k = \frac{\sqrt{2m\varepsilon}}{\hbar}
\]
\[
dk = \frac{\sqrt{2m}}{\hbar} \frac{1}{2} \varepsilon^{-\frac{1}{2}} d\varepsilon
\]
\[
\frac{k^{2}}{\pi^{2}} dk = \frac{1}{\pi^{2}} \frac{2m}{\hbar^{2}} \frac{\sqrt{2m}}{\hbar} \frac{1}{N_{\varepsilon}} d\varepsilon = g(\varepsilon) d\varepsilon
\]
\[
g(\varepsilon) = \frac{m}{\pi^{2} \hbar^{2} N \varepsilon}
\]
\[
g(\varepsilon) d\varepsilon = 2 \frac{1}{8\pi^{3}} \frac{1}{4\pi} k^{2} dk
\]
\[
= \left( \frac{1}{V} \right) X \left[ \text{the number of one-electron levels in the energy range from } \varepsilon \text{ to } \varepsilon + d\varepsilon \right]
\]
\[
g(\varepsilon) \text{ density of states}
\]
\[
\int \frac{d^{3}k}{4\pi^{3}} \tilde{F}(\varepsilon(\mathbf{k})) = \int_{0}^{v_{0}} d\varepsilon \ g(\varepsilon) \tilde{F}(\varepsilon)
\]
\[
= \int_{-\infty}^{v_{0}} d\varepsilon \ g(\varepsilon) \tilde{F}(\varepsilon)
\]
where \[
g(\varepsilon) = \frac{m}{\hbar^{2} \pi^{2} N} \varepsilon \frac{2m}{\hbar^{2}} \varepsilon > 0
\]
\[
= 0 \quad \varepsilon < 0
\]
\[\text{or} \quad g(\varepsilon) = \frac{3}{2} \frac{n}{\varepsilon_{F}} (\varepsilon / \varepsilon_{F}) \frac{1}{2}, \varepsilon > 0
\]
\[
= 0 \quad \varepsilon < 0
\]
where \[
n = \frac{3\pi^{2}}{2m}
\]
\[
\varepsilon_{F} = \frac{\hbar^{2}}{2m}
\]
A quantity of particular numerical importance: density of states at Fermi energy

\[ g(E_F) = \frac{m k_F}{\hbar^2 \pi^2} \]

or \[ g(E_F) = \frac{3}{2} \frac{n}{E_F} \]

Using this notation, rewrite \( u \) and \( n \).

\[ u = \int_{-\infty}^{\infty} d\varepsilon \, g(\varepsilon) \varepsilon f(\varepsilon) \]

\[ n = \int_{-\infty}^{\infty} d\varepsilon \, g(\varepsilon) f(\varepsilon) \]

\[ \text{at } T = 0 \]

\[ \text{at } T \approx 0.01 \frac{\mu}{k_B} \]

\[ \Delta \varepsilon \sim k_B T \]
In general, the integrals for \( u \) and \( n \) have a complex structure. But for \( T \ll T_F \), \( f \) differs from its zero temperature form only in a small region about \( \mu \) of width a few \( k_B T \).

Integrals of the form \( \int_{-\infty}^{\infty} H(\epsilon)f(\epsilon) \, d\epsilon \) differ from their zero temperature values \( \int_{-\infty}^{\epsilon_F} H(\epsilon) \, d\epsilon \) will be entirely determined by the form of \( H(\epsilon) \) near \( \epsilon = \mu \)

\[
\int_{-\infty}^{\infty} \, d\epsilon \, H(\epsilon) \, f(\epsilon) \quad \text{or} \quad f(\epsilon) = \frac{1}{\epsilon e^{(\epsilon - \mu)/k_B T} + 1}
\]

where \( H(\epsilon) \) vanishes as \( \epsilon \to -\infty \) and diverges no more rapidly than some power of \( \epsilon \) as \( \epsilon \to +\infty \).

Define \( k(\epsilon) = \int_{-\infty}^{\epsilon} H(\epsilon') \, d\epsilon' \)

So that \( H(\epsilon) = \frac{d}{d\epsilon} k(\epsilon) \)

Integrate by parts

\[
\int_{-\infty}^{\infty} H(\epsilon) f(\epsilon) \, d\epsilon = k(\epsilon) f(\epsilon) \bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} k(\epsilon) \left( -\frac{\partial f}{\partial \epsilon} \right) \, d\epsilon
\]
Provided that $H$ is non-singular and not too rapidly varying around $E = \mu$, it is reasonable to expand $k(E)$ in a Taylor series about $E = \mu$

$$k(E) = k(\mu) + \sum_{n=1}^{m} \left[ \frac{(E-\mu)^n}{n!} \right] \left[ \frac{d^n k(E)}{dE^n} \right]_{E=\mu}$$

Note that $\int_{-\infty}^{\infty} \left( -\frac{\partial f}{\partial E} \right) dE = 1$

$$\int_{-\infty}^{\infty} H(E) f(E) dE = \int_{-\infty}^{\mu} H(E) dE + \sum_{n=1}^{\infty} \int_{-\infty}^{\mu} \frac{(E-\mu)^{2n}}{(2n)!} \left( -\frac{\partial f}{\partial E} \right) dE \frac{d^{2n-1}}{dE^{2n-1}} H(E) \left|_{E=\mu} \right.$$ 

$(\frac{\partial f}{\partial E})$ is even function of $(E-\mu)$, only even terms in $(\ast)$ contribute.

Making substitution $\frac{(E-\mu)}{k_B T} = x$

$$\int_{-\infty}^{\infty} H(E) f(E) dE = \int_{-\infty}^{\mu} H(x) dE + \sum_{n=1}^{\infty} A_n \left( k_B T \right)^{2n} \frac{d^{2n-1}}{dE^{2n-1}} H(E) \left|_{E=\mu} \right.$$ 

$$A_n = \int_{-\infty}^{\infty} \frac{x^{2n}}{(2n)!} \left( -\frac{1}{1+x} \right) dx$$

$$A_n = 2 \left( 1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \frac{1}{5^{2n}} - \cdots \right)$$

Sommerfeld expansion

$$a_1 = \frac{\pi^2}{6}, \quad a_2 = \frac{7\pi^4}{360}$$
\[ \sum_{\infty}^{\infty} H(\varepsilon) f(\varepsilon) d\varepsilon = \sum_{-\infty}^{\infty} H(\varepsilon) d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 H'(\mu) + \frac{\pi^4}{360} (k_B T)^4 H''''(\mu) + \ldots \]

To evaluate specific heat, apply Sommerfeld expansion to electronic energy and number density:

\[ U = \int_{0}^{\mu} \varepsilon g(\varepsilon) d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 \left[ \mu g'(\mu) + g(\mu) \right] + O(T^4) \]

\[ n = \int_{0}^{\mu} g(\varepsilon) d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 g'(\mu) + O(T^4) = \int_{0}^{\varepsilon_F} g(\varepsilon) d\varepsilon \]

implies that \( \mu \) differs from \( \varepsilon_F \) by terms of order \( T^2 \).

Correctly to the order of \( T^2 \):

\[ \int_{0}^{\mu} H(\varepsilon) d\varepsilon = \int_{0}^{\varepsilon_F} H(\varepsilon) d\varepsilon + (\mu - \varepsilon_F) H(\varepsilon_F) \]

So,

\[ U = \int_{0}^{\varepsilon_F} \varepsilon g(\varepsilon) d\varepsilon + \varepsilon_F \left\{ (\mu - \varepsilon_F) g(\varepsilon_F) + \frac{\pi^2}{6} (k_B T)^2 g'(\varepsilon_F) \right\} \]

\[ + \frac{\pi^2}{6} (k_B T)^2 g(\varepsilon_F) + O(T^4) \]

\[ n = \int_{0}^{\varepsilon_F} g(\varepsilon) d\varepsilon + \left\{ (\mu - \varepsilon_F) g(\varepsilon_F) + \frac{\pi^2}{6} (k_B T)^2 g'(\varepsilon_F) \right\} \]

\[ 0 = (\mu - \varepsilon_F) g(\varepsilon_F) + \frac{\pi^2}{6} (k_B T)^2 g'(\varepsilon_F) \]
\[ \mu = \varepsilon_F - \frac{\pi^2 (k_B T)^2}{6} \frac{g'(\varepsilon_F)}{g(\varepsilon_F)} \]

\[ g(\varepsilon) = \frac{3}{2} \frac{n}{\varepsilon_F} \left( \frac{\varepsilon}{\varepsilon_F} \right)^{1/2} \]

\[ \mu = \varepsilon_F \left[ 1 - \frac{1}{3} \left( \frac{\pi (k_B T)}{2 \varepsilon_F} \right)^2 \right] \]

\[ u = u_0 + \frac{\pi^2}{6} (k_B T)^2 g(\varepsilon_F) \]

\[ C_V = (\frac{\partial u}{\partial T})_n = \frac{\pi^2}{3} k_B^2 T g(\varepsilon_F) \]

\[ g(\varepsilon_F) = \frac{3}{2} \frac{n}{\varepsilon_F} \quad \text{for free electron} \]

\[ C_V = \frac{\pi^2}{2} \left( \frac{k_B T}{\varepsilon_F} \right)_n k_B \quad \text{for free electron} \]
Electrical Conductivity of Free Electron
Fermi Gas

Consider the dynamics of $N$ electron non-interactivity. Fermi gas in external electromagnetic field. The statistics of the electrons is profoundly affected by quantum mechanics. But the dynamical behavior can be determined by considering $N$-independent one-electron problems using classical mechanics.

For one single electron:
Size of the wave packet ($\Delta x$): $y_s \sim (2-6)\alpha_0 \sim$ a few Å

$\Delta x \ll \lambda \quad \rightarrow$ wavelength of external field.

Electron wave packets can be treated as classical particles. The $N$-electron state can be determined by tracing the time evolution of the $N$ single-electron states.

One electron in a electric field $\vec{E}$

$m \ddot{\vec{r}} = \hbar \vec{E}$.

$\vec{F} = m \frac{d\vec{v}}{dt} = -e(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B}) \quad \vec{B} = 0$

$\frac{\hbar}{2} \frac{d\vec{k}}{dt} = -e \vec{E}$.

$\frac{d\vec{k}}{dt} = -\frac{e}{\hbar} \vec{E}$
\[ \mathbf{k}(t) - \mathbf{k}(0) = -\frac{eE}{\hbar} t. \]

A \( \mathbf{k} \) vector will experience linear motion with constant speed.

For \( N \) electrons

\[ \mathbf{j} = n(-e) \sum \mathbf{v} = 0 \]

\( t = 0 \): Fermi sphere centered.

At \( t = 0 \): No net momentum.

At \( t \), Fermi sphere is shifted by \( \delta k = -\frac{eE}{\hbar} t \).

However, the linear motion in \( \mathbf{k} \) space will be compromised by scatterings from impurities, defects, and phonons.

If collision time (relaxation time) is \( \tau \), the Fermi sphere will be off-centered by \( \frac{eE}{\hbar} \tau \).
Scattering across Fermi sphere

$V_F \sim 10^8 \text{cm/s}$

$\Delta V \sim 0.1 \text{ cm/s}.$

Elections at Fermi Surface:
Contribute to conduction.

$\Delta \vec{k} = -\frac{e \vec{E} \tau}{\hbar}$

$\Delta \vec{v} = -\frac{e \vec{E} \tau}{m}$

$\vec{j} = n(e) \Sigma \vec{v} + \Delta \vec{v}.$

$= -ne \Sigma \vec{v} + n(e) \Sigma \Delta \vec{v}.$

$\vec{j} = \frac{ne^2 \vec{E} \tau}{m}.$

$\sigma = \frac{ne^2 \tau}{m}.$

Mean free path: $l = V_F \cdot \tau.$

at RT $l > 100 \text{Å} \gg r_s.$

Further justify treating electrons as classical particles.
Thermal conductivity of free electron Fermi gas.

\[ \kappa = \frac{1}{3} \nu^2 c \sigma n \]

Drude: \( \frac{1}{2} m \nu^2 = k_B T \)
\[ \nu^2 = \frac{2k_B T}{\pi m} \]

Sommerfeld: \( \frac{1}{2} m \nu^2 = k_B T_F = \varepsilon_F \)
\[ \nu^2 = \frac{2k_B T_F}{m} \]

Drude's estimate of \( \nu^2 \) is too small by factor \( \frac{T}{T_F} \sim 0.01 \)
Drude's estimate of \( C_v \) is too large by \( \frac{T_F}{T} \sim 10^2 \)

\[ C_v = \frac{3}{2} n k_B \]

\[ C_v = \frac{\pi^2}{2} \left( \frac{k_B T}{\varepsilon_F} \right) n k_B. \]