

PHYS 813: Statistical Mechanics, Assignment 9

Due 5/08/08

1. Consider a Bose gas of photons in a volume V and temperature T . The photon energy is $\epsilon = pc$ where p is the photon momentum and c is the speed of light. For photons, the chemical potential μ is always $\mu = 0$. The number of photons is not well determined since the number of photons at $\epsilon = 0$ cannot be defined (cf. the BE expression). Assume that the initial temperature is high and there are initially N photons with nonzero energies.

- (a) Find an expression for the maximum value of N at a given T .
- (b) For a given N , the lowest temperature “supporting” this number can be loosely called T_c , as if the temperature is lowered below T_c , some photons “condense”. Calculate the pressure p and the energy U for $T < T_c$. Is the classical relation $p = U/3V$ preserved?

2. Consider a noninteracting Bose gas in which the lowest excited state is separated from the ground state by a gap Δ , i.e.,

$$\epsilon(\mathbf{p}) = \begin{cases} 0 & p = 0 \\ \Delta + p^2/2m & p \neq 0 \end{cases}$$

where $p = |\mathbf{p}|$ and m is the mass of a particle. We can assume that the spectrum above the gap is quasicontinuous. Find the temperature dependence of the specific heat C_V for low temperatures T , i.e., for $kT \ll \Delta$, where k is the Boltzmann constant.

3. Atoms in solids can exhibit various collective vibrational motions. The normal frequency ω of a given vibrational mode is connected with the wave number k of a corresponding plane-wave motion by the so-called dispersion relation $\omega = Ak^s$, where A is a constant. For the best-known phonon waves, the relation is $\omega = vk$, where v is the velocity of sound in the solid. Show that the vibrational motion contribution to the specific heat of a solid at low temperatures T is proportional to $T^{3/s}$. For phonons, this dependence is called the Debye’s T^3 law. The case of $s = 2$ corresponds to the spin waves propagating in ferromagnetic systems. Use Debye’s assumption that the frequencies are distributed continuously from zero to a cutoff value ω_D and that the density of states in k is $g(k) = 12\pi V k^2 / (2\pi)^3$.
4. Despite its simplicity, the free-electron model of the electron conduction in metals predicts various properties quite successfully. Use this model to calculate the isothermal compressibility of a metal as a function of the electron number density n and the Fermi energy ϵ_F at $T = 0$ K.
5. Consider a degenerate (assume here $T = 0$), relativistic (i.e., the relation between particle’s energy ϵ and momentum p is $\epsilon = pc$, where c is the speed of light) gas of fermions of spin $\frac{1}{2}$.
 - (a) Find the expression for the total energy of the system as a function of the expectation number of particles N and the total volume V .
 - (b) Calculate the pressure in this systems.
6. Suppose that in some sample the density of states of electrons $g(\epsilon)$ is a constant g_0 for energies $\epsilon > 0$ ($g(\epsilon) = 0$ for $\epsilon < 0$) and the total number of electrons is N .
 - (a) Calculate the Fermi energy ϵ_F at $T = 0$ K.
 - (b) For nonzero temperatures, derive the condition that the system is nondegenerate. Express your answer in terms of the relation between kT and ϵ_F .
 - (c) Show that the electronic specific heat is proportional to T when the system is nearly degenerate. Use simple physical arguments rather than elaborate derivations.

7. Since in the Fermi-Dirac statistics the grand partition functions for ideal gas is given by $\mathcal{Z}(V, T, \mu) = \prod_i (1 + ze^{-\beta\epsilon_i})$, where $\beta = 1/kT$, ϵ_i are single-particle energy levels, μ is the chemical potential, and $z = e^{\mu/kT}$, this form does not exhibit a direct relation to the macroscopic parameters of the gas, as it is the case for its classical counterpart. Show that the relation

$$q(V, T, \mu) = \ln \mathcal{Z}(V, T, \mu) = \frac{pV}{kT} \quad (1)$$

does hold for the Fermi-Dirac ideal gas, proceeding as follows:

- (a) Using the relations $q = (gV/\lambda^3)f_{5/2}(z)$ and $N = (gV/\lambda^3)f_{3/2}(z)$, where g is the spin degeneracy, $\lambda = h/\sqrt{2\pi mkT}$, and

$$f_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1} dx}{z^{-1}e^x + 1},$$

with $\Gamma(n) = (n-1)(n-2)\cdots\frac{1}{2}\sqrt{\pi}$ for half integer arguments, show that

$$N = z \left(\frac{\partial q}{\partial z} \right)_{V, T}.$$

- (b) Starting from $U = \sum n_i \epsilon_i$, where $n_i = 1/(z^{-1}e^{\beta\epsilon_i} + 1)$ is the occupation number for state ϵ_i , derive an expression for U which can be related to q .
- (c) Show that the differential of $q(V, T, \mu)$ can be written as $dq = -Ud\beta - \beta dW - Nd\alpha$, where W is work performed on the gas and $\alpha = -\ln z$.
- (d) Show that this relation implies that the entropy $S = k(q + \alpha N + \beta U)$.
- (e) Combine the derived relation with the Euler formula to get Eq. (1).