PHYS 813: Statistical Mechanics, Assignment 4

Due 3/20/08

1. Consider a quantum free particle of mass m in a cubic box of volume V (with the wave function equal to zero on the walls). Show that the number of states with energies below a given value E is

$$\Sigma_1 = \frac{\pi}{6}\varepsilon^{3/2} - \frac{3\pi}{8}\varepsilon + \cdots$$

where $\varepsilon = 8mV^{2/3}E/h^2$. *Hint:* The second term in the equation above is due to "surface" effects neglected when cutting out the 1/8 of a sphere to get the first term. Consider the whole sphere to account for the surface effects corresponding to some quantum numbers equal to zero.

2. Consider an extreme relativistic gas in a cubic box. The single-particle energy states for such a gas are given by

$$E_{n_x,n_y,n_z} = \frac{hc}{2L}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

where h is the Planck constant, c is the speed of light, and L is the dimension of the box. Using arguments similar to those for the nonrelativistic case, show that the ratio of $C_p/C_V = 4/3$ for such gas. *Hint:* There is no need to explictly evaluate the number of states to answer this problem.

3. Consider the entropy of mixing ΔS of two gases, intitially at the same temperature (use the expressions with the Gibbs correction factor). Analyze this quantity for a fixed total volume V and particle number N. Consider both the case of different and identical particles.

   (a) Assume that the initial numbers of particles are constant and show that the entropy of mixing is at the minimum for equal particle densities. Sketch ΔS using a contour plot to visualize your findings.

   (b) Assume equal particle densities before mixing and show that

   $$\Delta S \leq Nk \ln 2$$

   and that the equality holds only for equal numbers of different particles.

4. Now assume that the initial temperatures of the two gases are different before mixing. Derive an expression for the entropy of mixing in this case. Does the contribution from this cause depend on the gases being identical or different?

5. Consider the one-dimensional harmonic oscillator.

   (a) For the quantum oscillator, with energy levels $\epsilon_n = (n + \frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$, derive an expression for the number of ways in which a given energy E can be distributed among a set of N oscillators.

   (b) Find the asymptotic form of this expression in the case when the number of quanta available $R = (E - \frac{1}{2}N\hbar\omega)/\hbar\omega$ is much larger than N. Next, find the number of microstates in the interval $[E, E + \Delta E]$.

(c) For the corresponding classical harmonic oscillator, find the expression for the "volume" of the region of the phase space accessible for this system. Compare with the expression obtained in (b) and find the "conversion factor" between the two results. You may need the formula for the volume of an N-dimensional sphere of radius R: $V_n = \pi^{n/2} R^n / (n/2)!$